On Automatically Proving the Correctness of `math.h` Implementations

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[1] Stanford University [2] KAIST [3] Microsoft Research

POPL 2018

Our Goal

$$log\, x$$

mathematical
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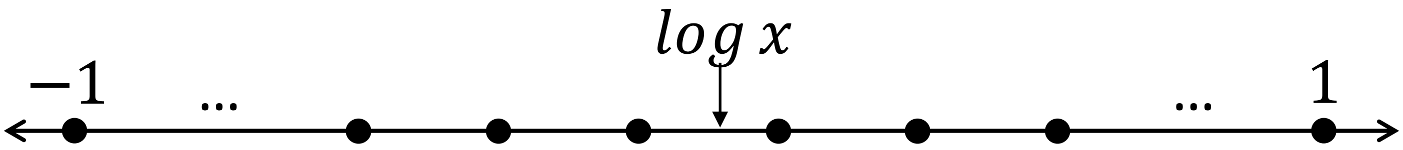

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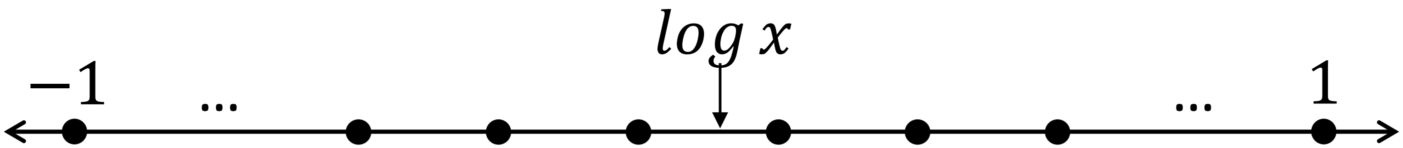

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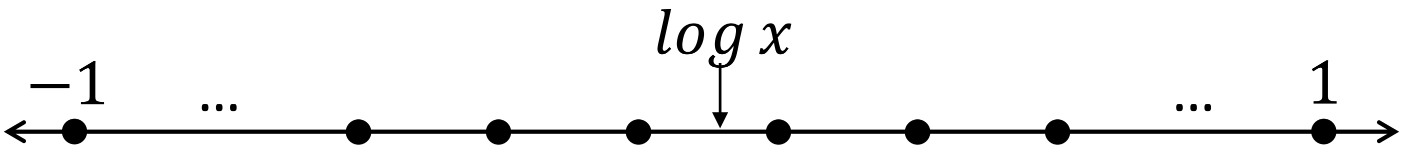

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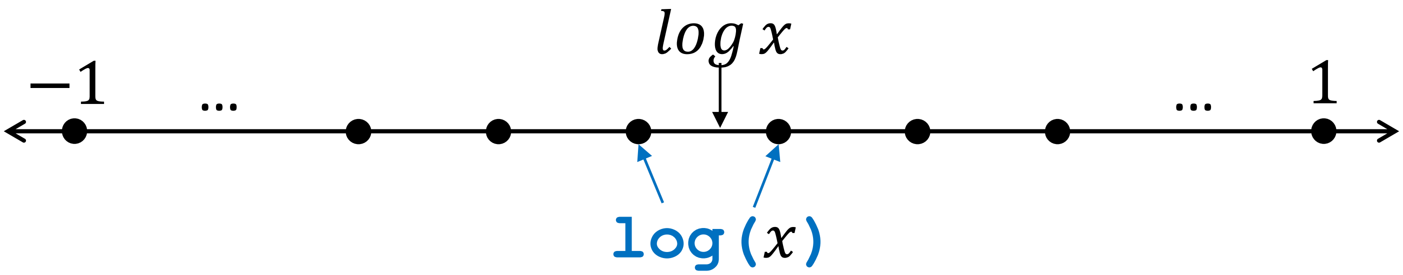

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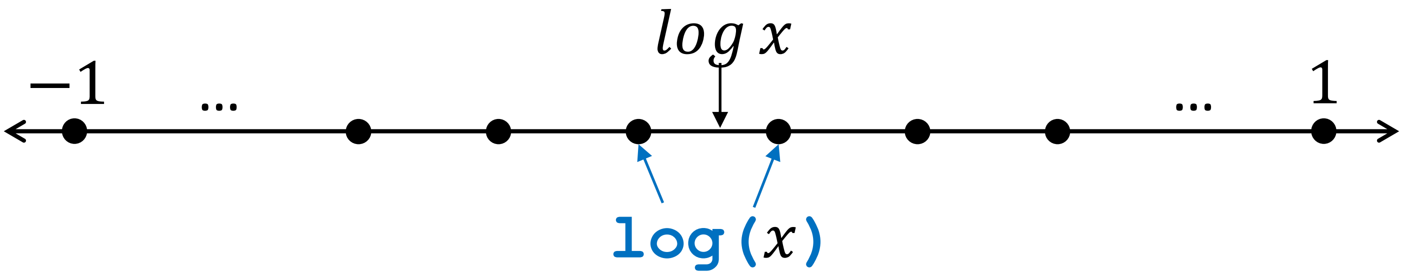

Goal: Prove <u>this claim</u> automatically!

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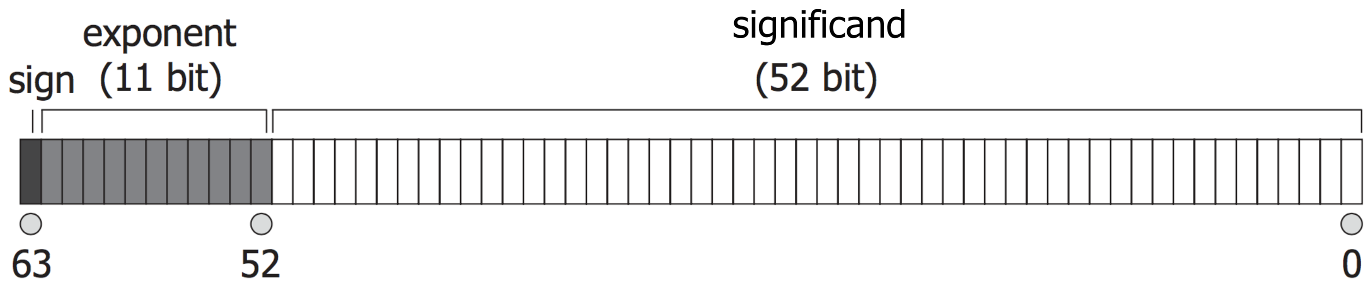
Floating-Point Numbers/Operations



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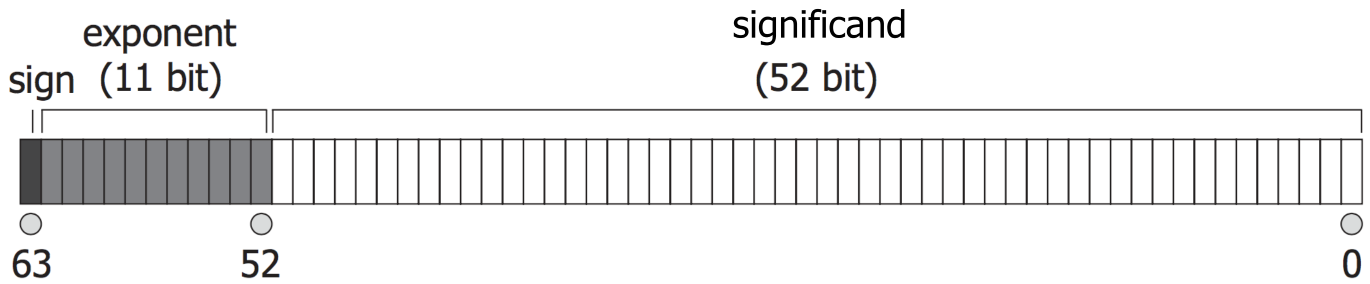

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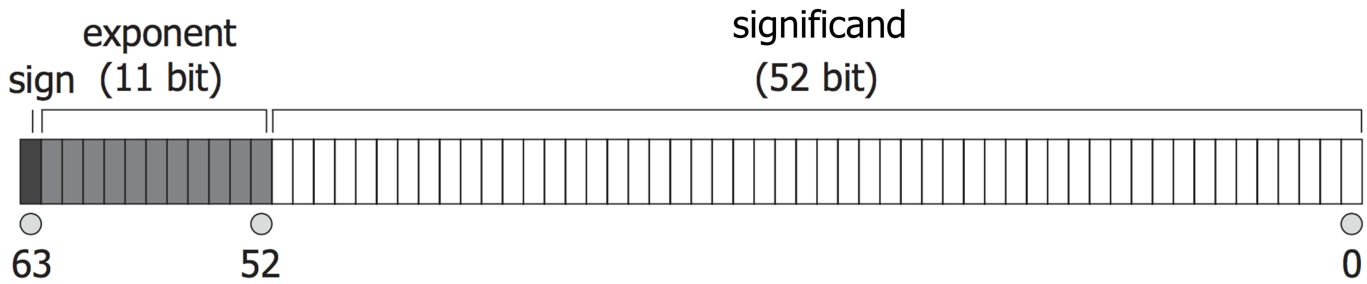

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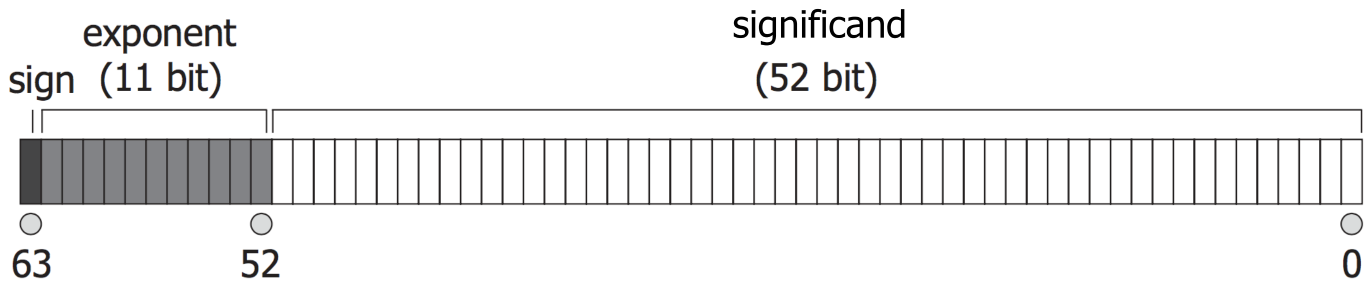

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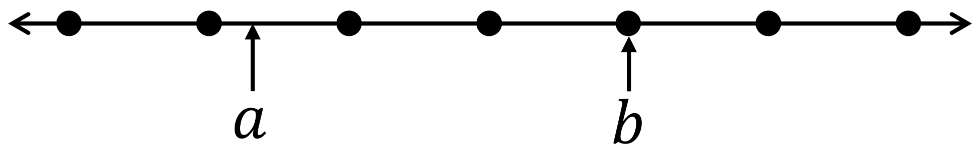

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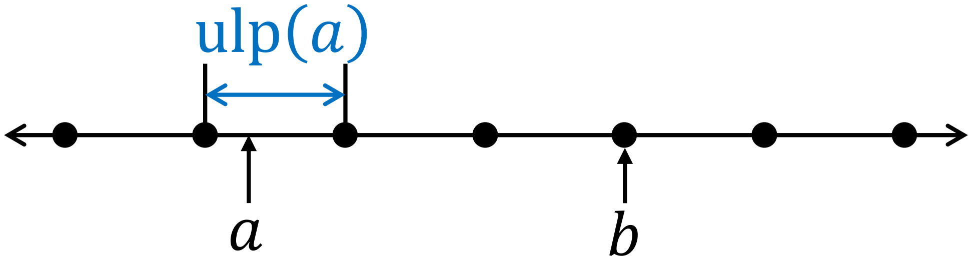

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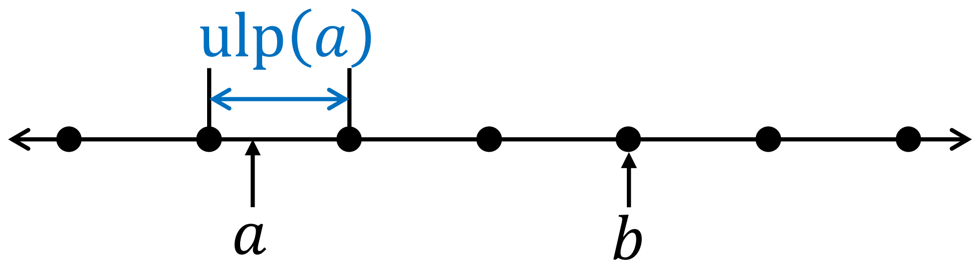

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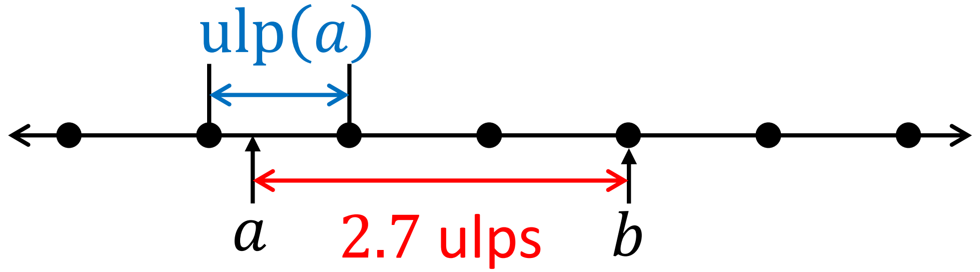

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Goal: want to find $\Theta < 1$

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  - None of them can prove < 1 ulp error bound automatically

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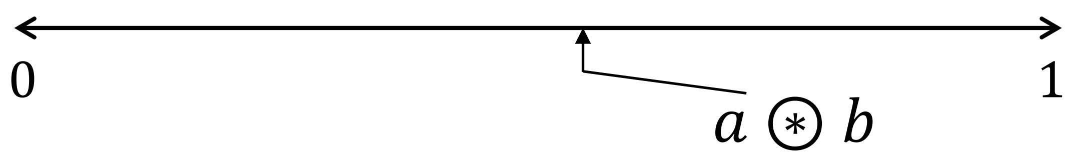

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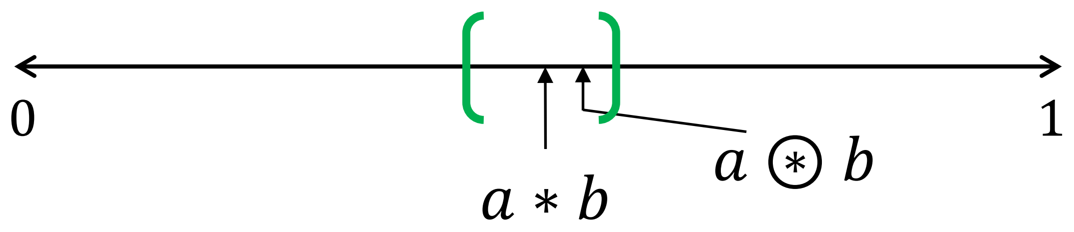

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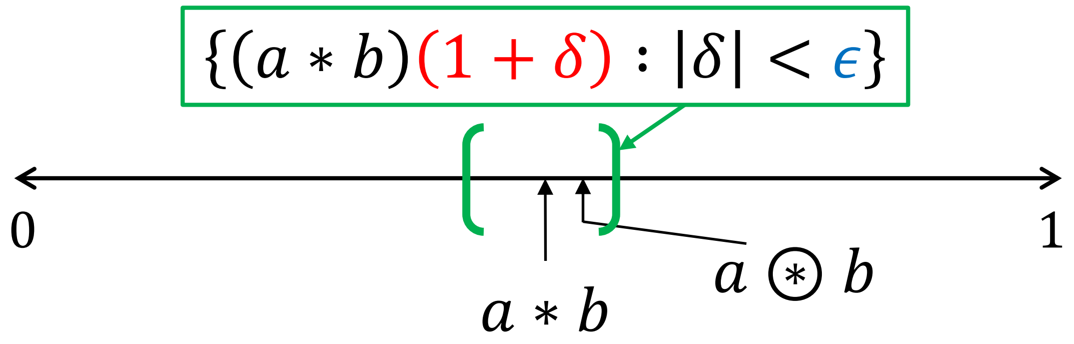

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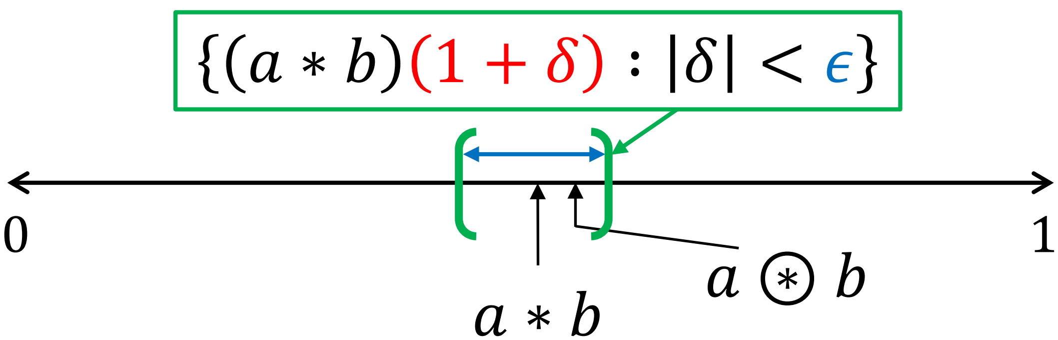

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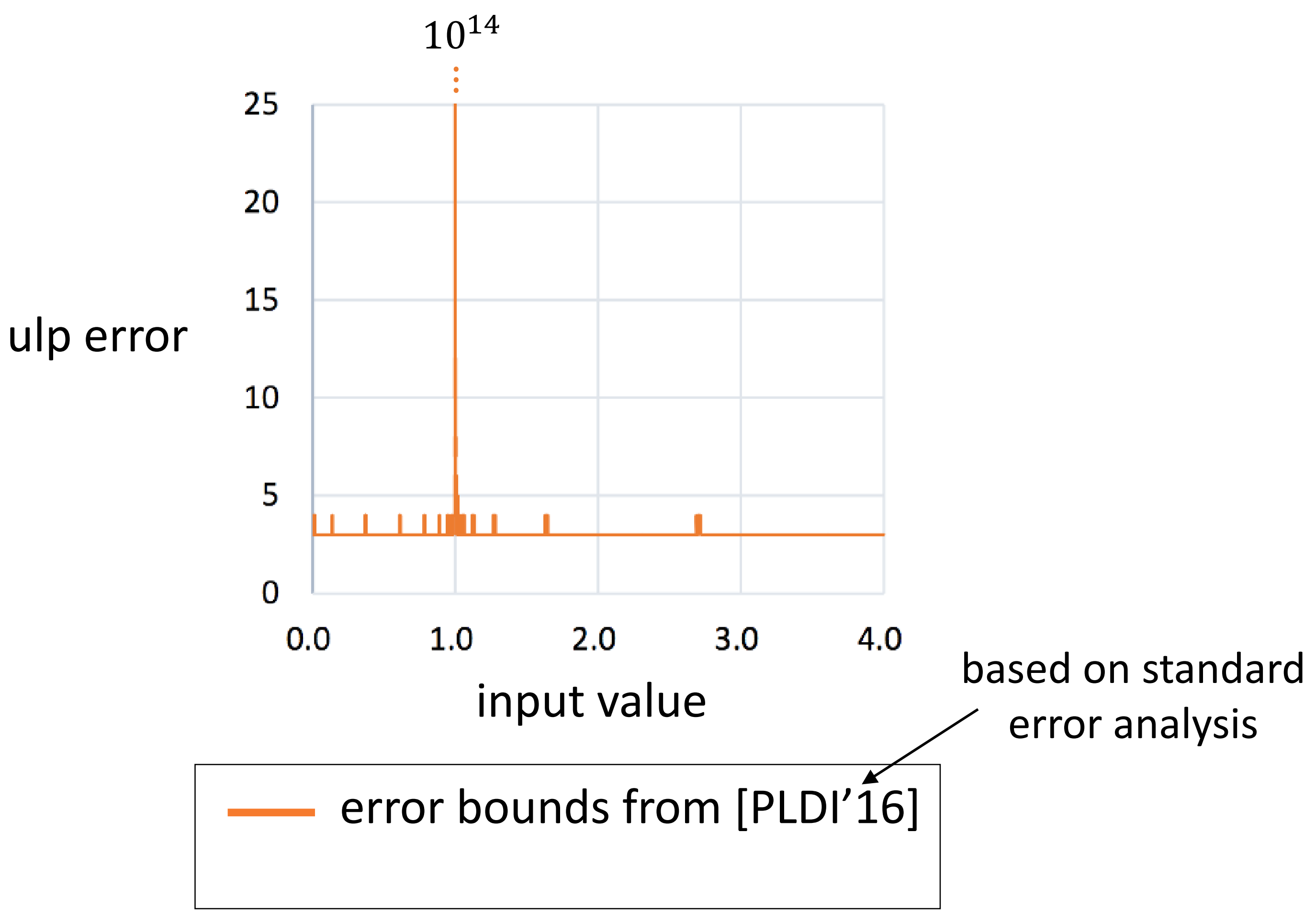

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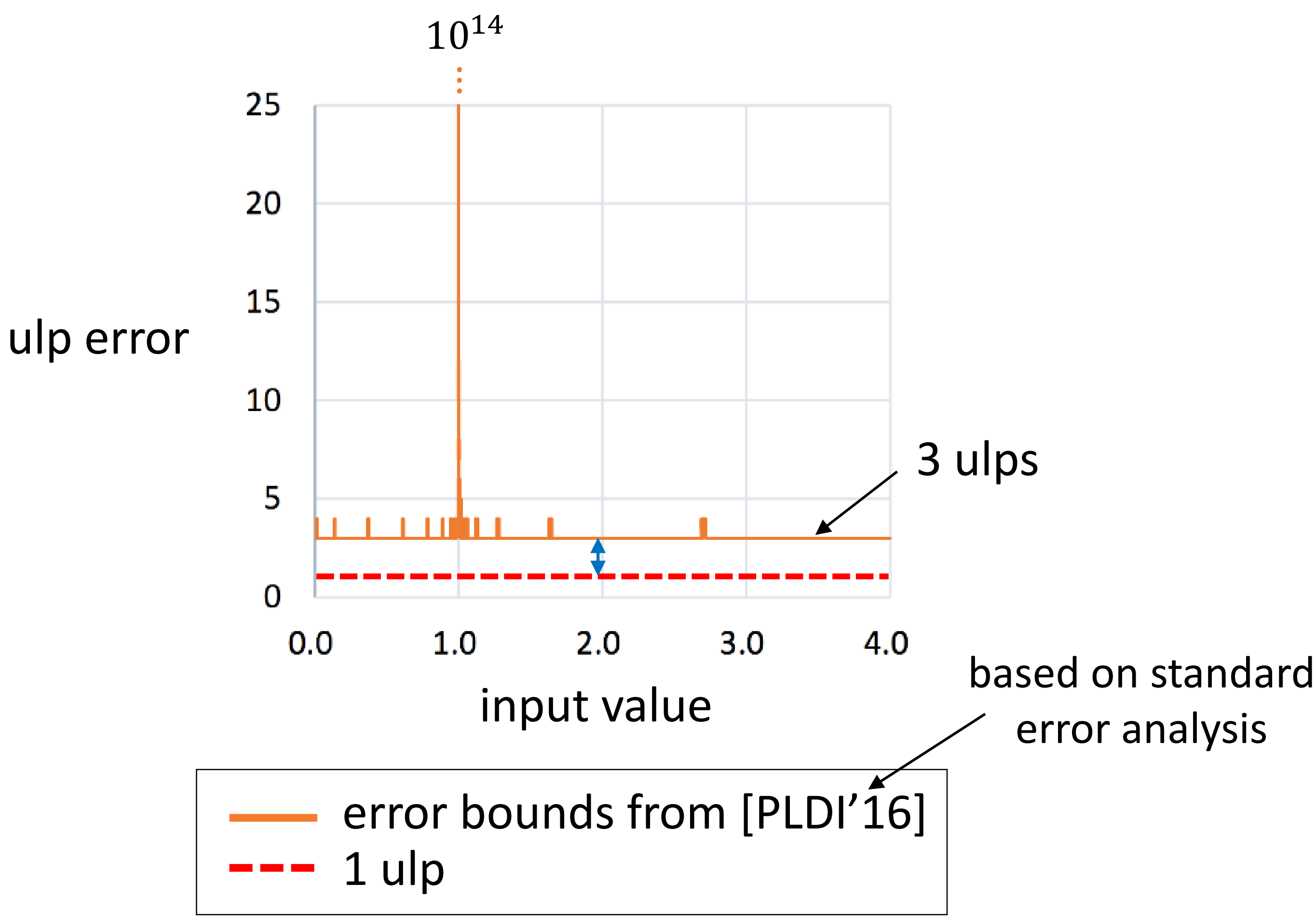

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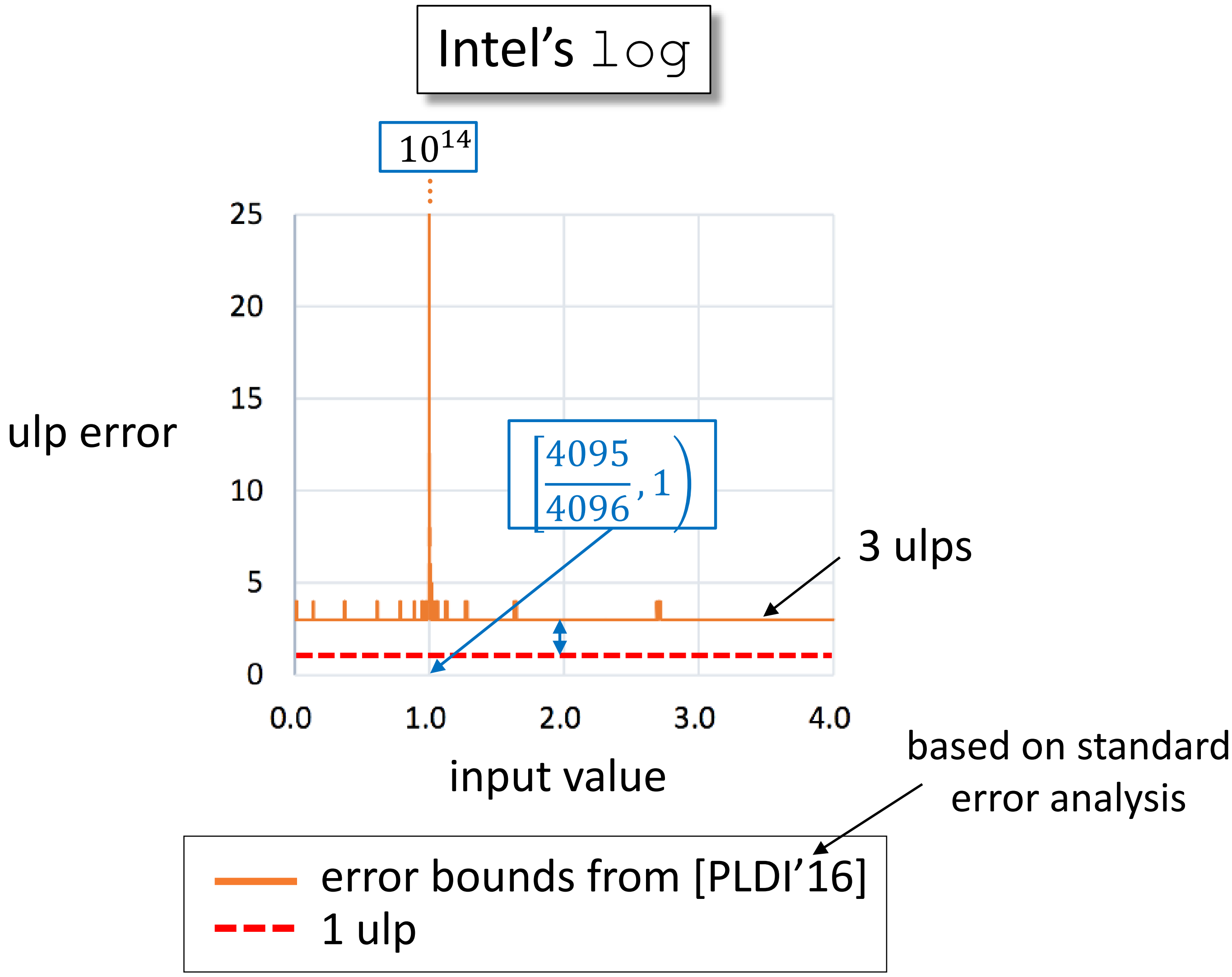

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$$0.125 \otimes x \sqsubseteq (0.125 \times x)(1 + \delta)$$

$$x \ominus 1 \sqsubseteq x - 1 \quad \text{if } 0.5 \leq x \leq 2$$

$$x \ominus 1 \sqsubseteq (x - 1)(1 + \delta')$$

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$$r(x) = \left[\left((2 \otimes x) \ominus \frac{255}{128}\right) \otimes \frac{1}{2}\right] \oplus \left[\left(\frac{255}{128} \otimes \frac{1}{2}\right) \ominus 1\right] \quad (\approx x - 1)$$

and returns

$$r(x) - \frac{1}{2} r(x)^2 + \cdots + \frac{1}{7} r(x)^7$$

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10^{14} ulps for `log` near $x = 1$ [PLDI'16]

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We prove a bound of 0.583 ulps for `log`

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To prove the 1 ulp error bound,

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To prove the 1 ulp error bound,

- Need to construct more precise abstractions
- Need to use floating-point exactness results

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- <u>Theorem</u> [Sterbenz, 1973]

$$\frac{1}{2}a \leq b \leq 2a \quad \Longrightarrow \quad a \ominus b = a - b$$

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  1. Compute $y = x \ominus 2\pi$
  2. Return $y - \frac{1}{3!}y^3 + \cdots + \frac{1}{9!}y^9 \ (\approx sin\, y)$

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precision loss of `sin` is small

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  - Naïve ways: $(a \oplus b) \oplus c$, $a \oplus (b \oplus c)$, $\cdots$
  - More accurate way:
    1. Compute $r = a \oplus b \ominus a \ominus b$
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Dekker's Theorem

- Theorem [Dekker, 1971]

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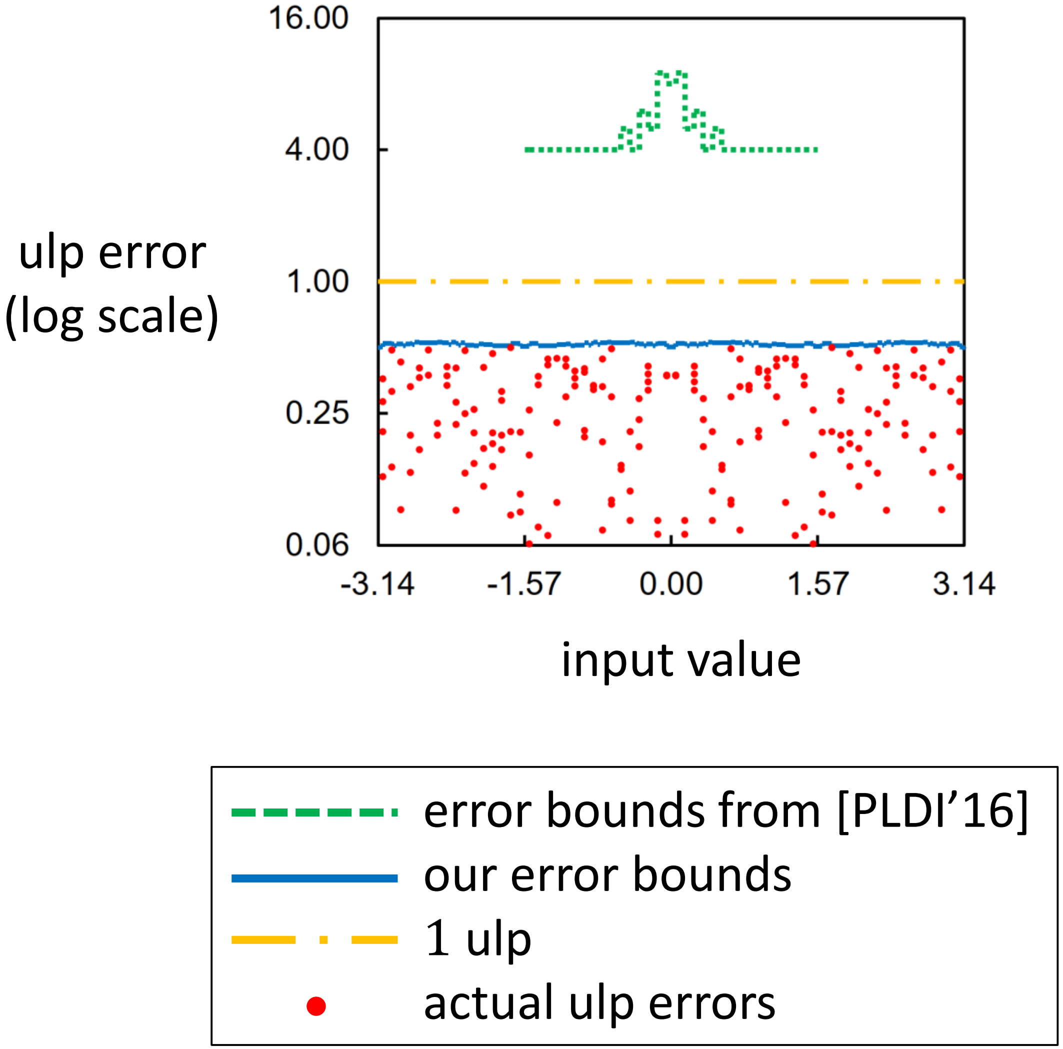
- Apply our technique to the resulting floating-point expressions
  - Use Mathematica to solve optimization problems analytically

Results: `sin` on $[-\pi, \pi]$



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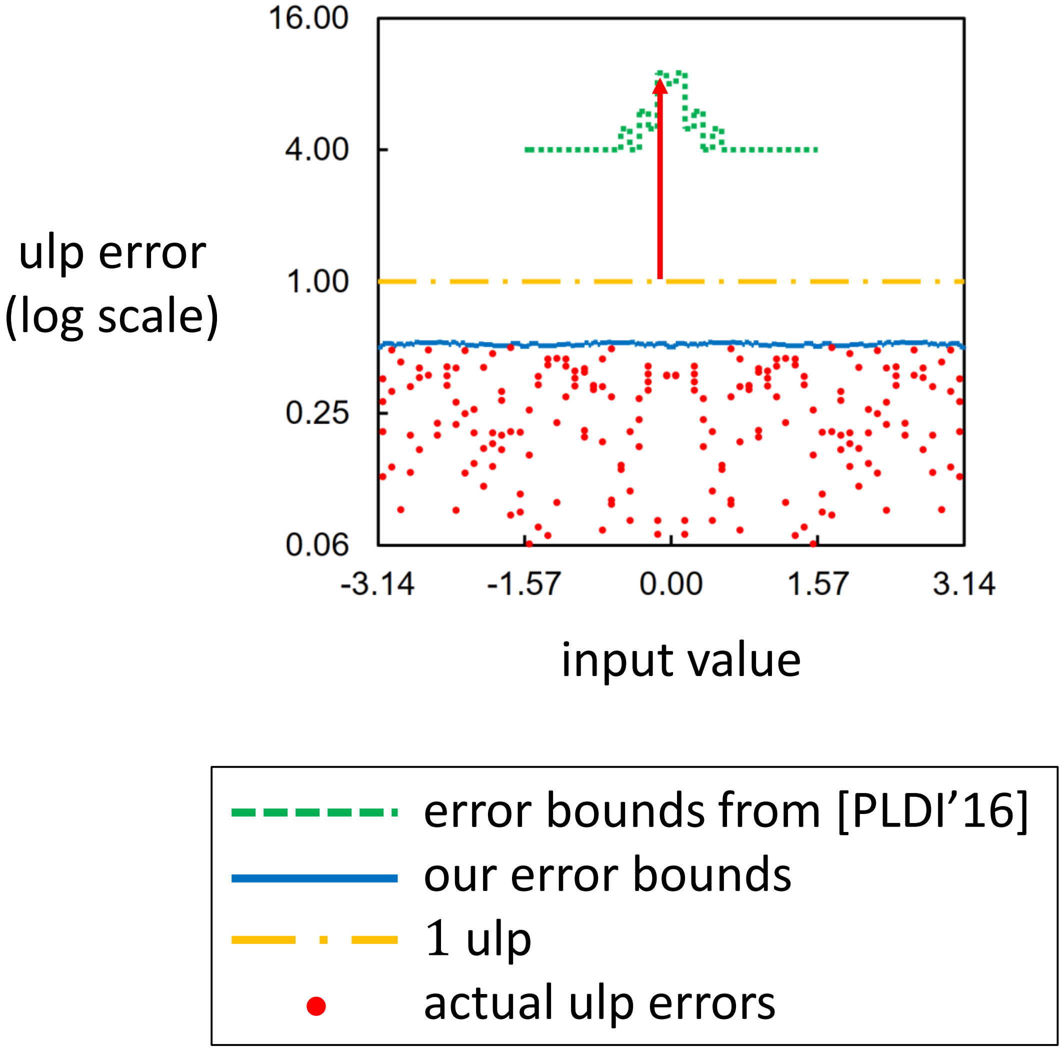

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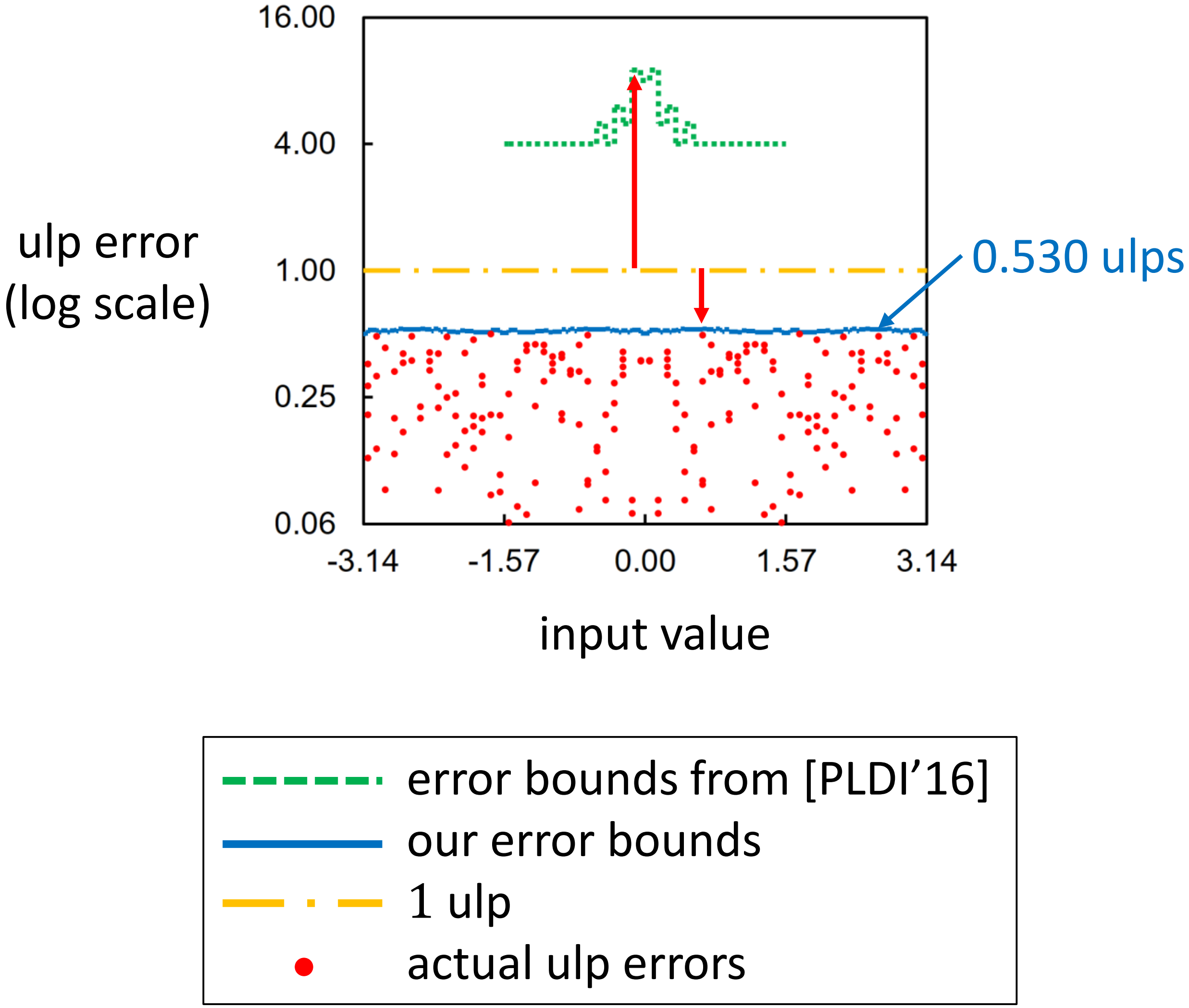

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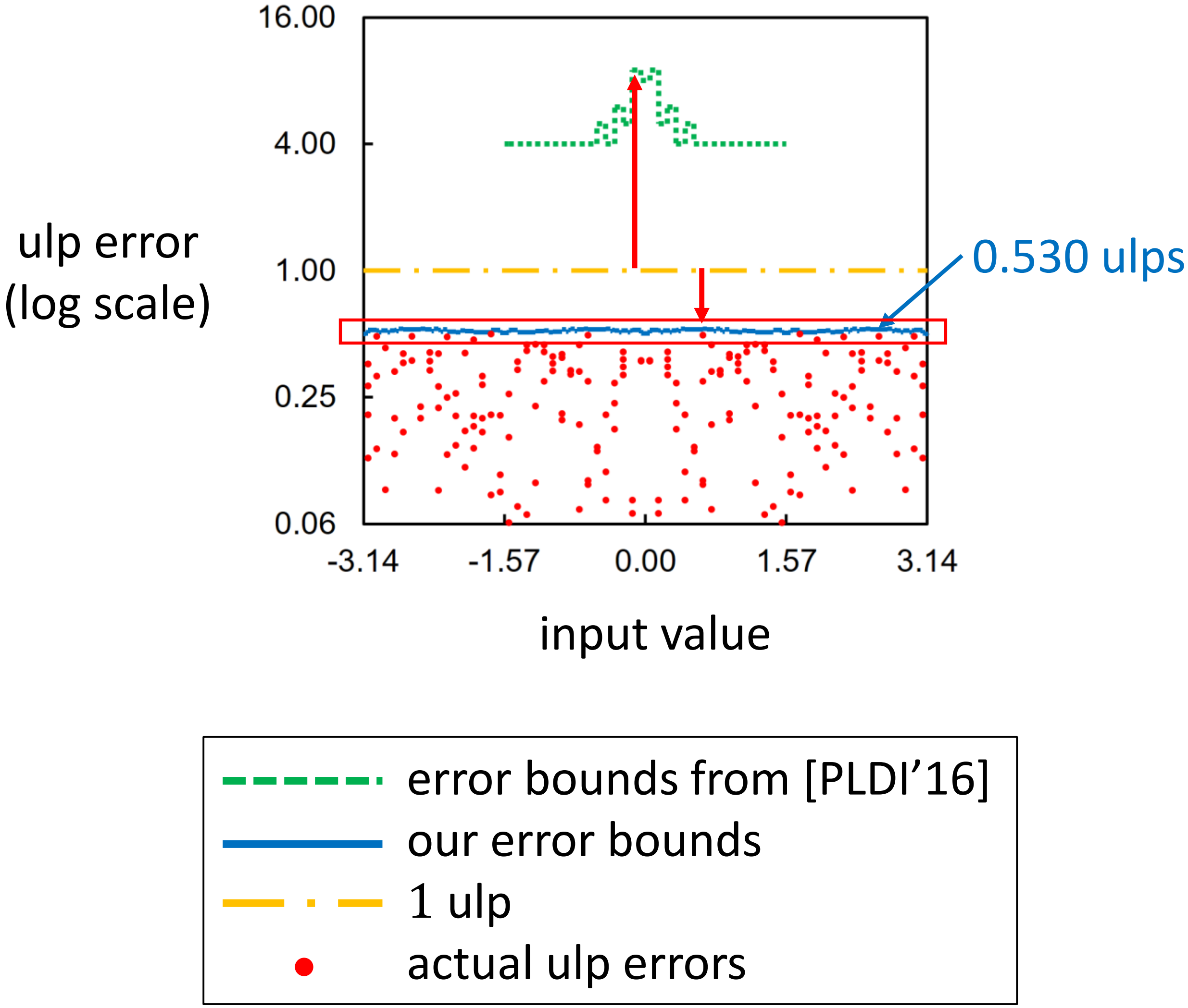

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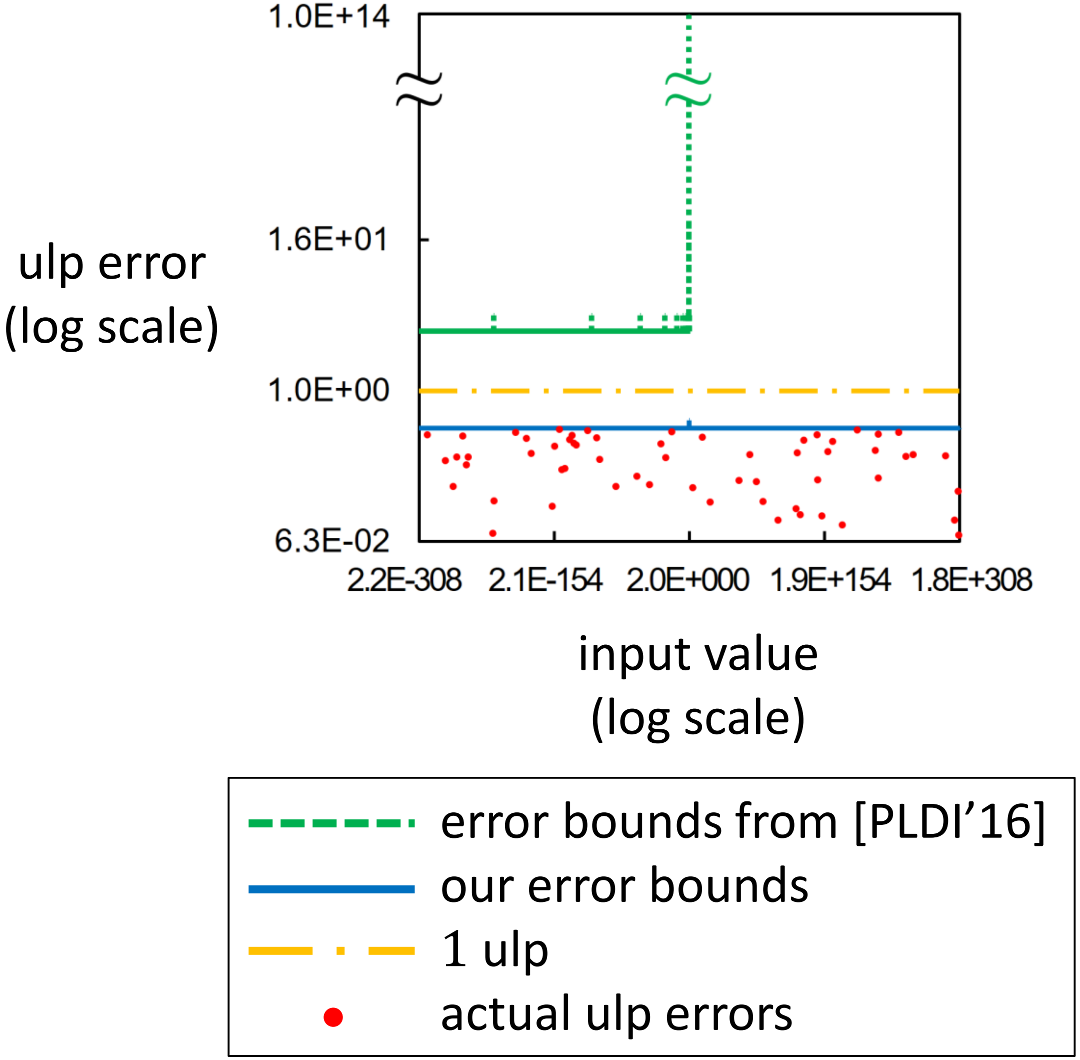

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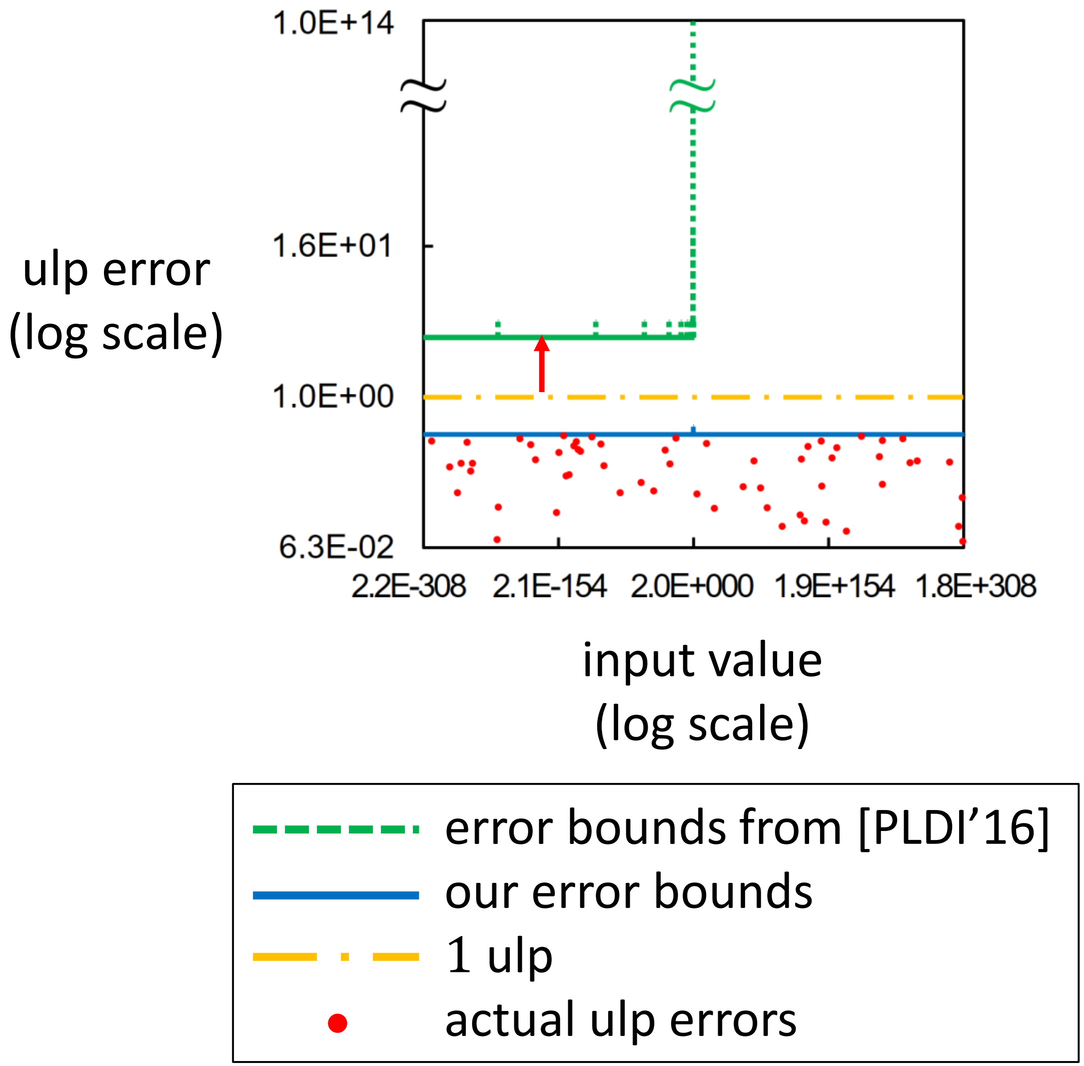

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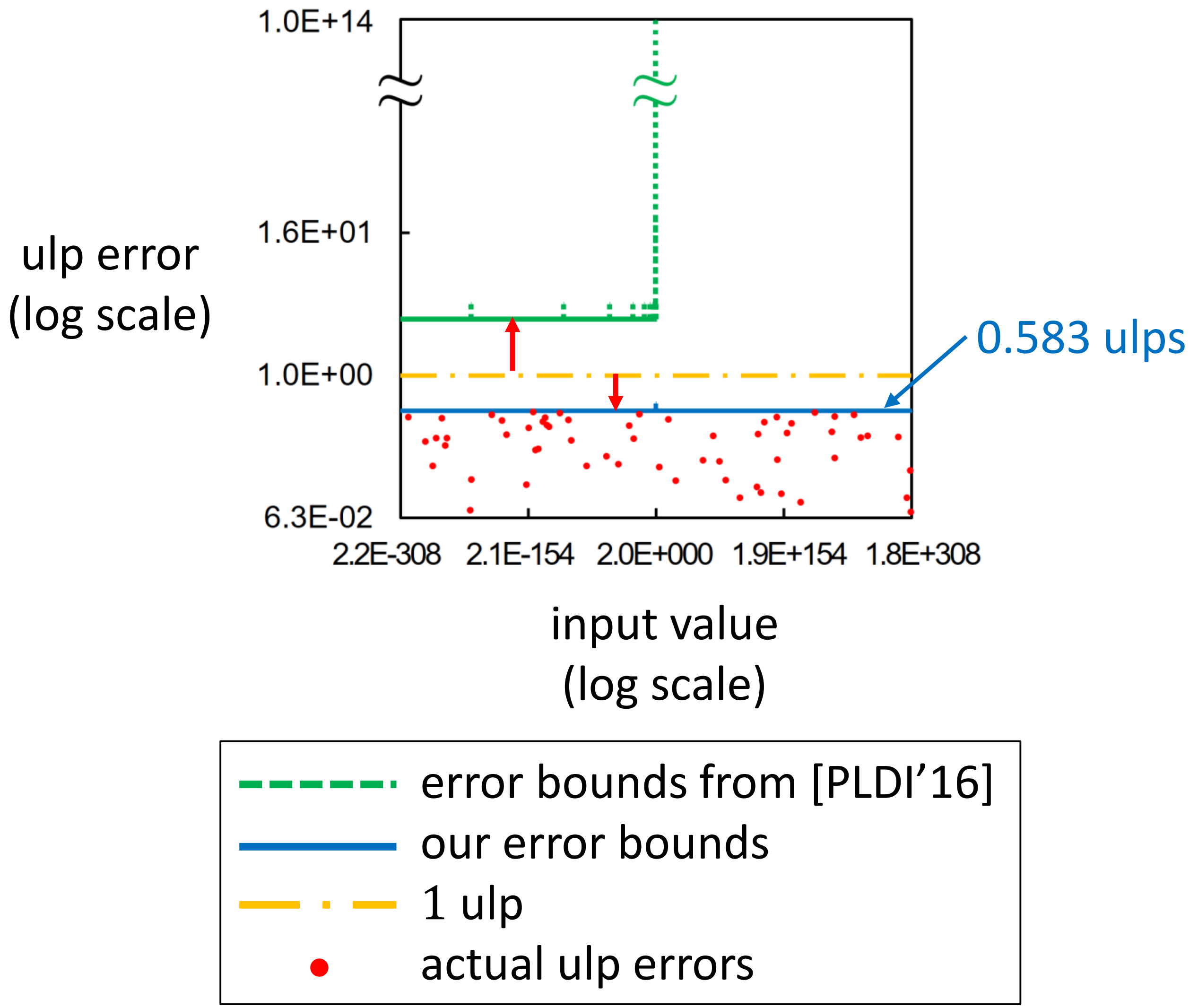

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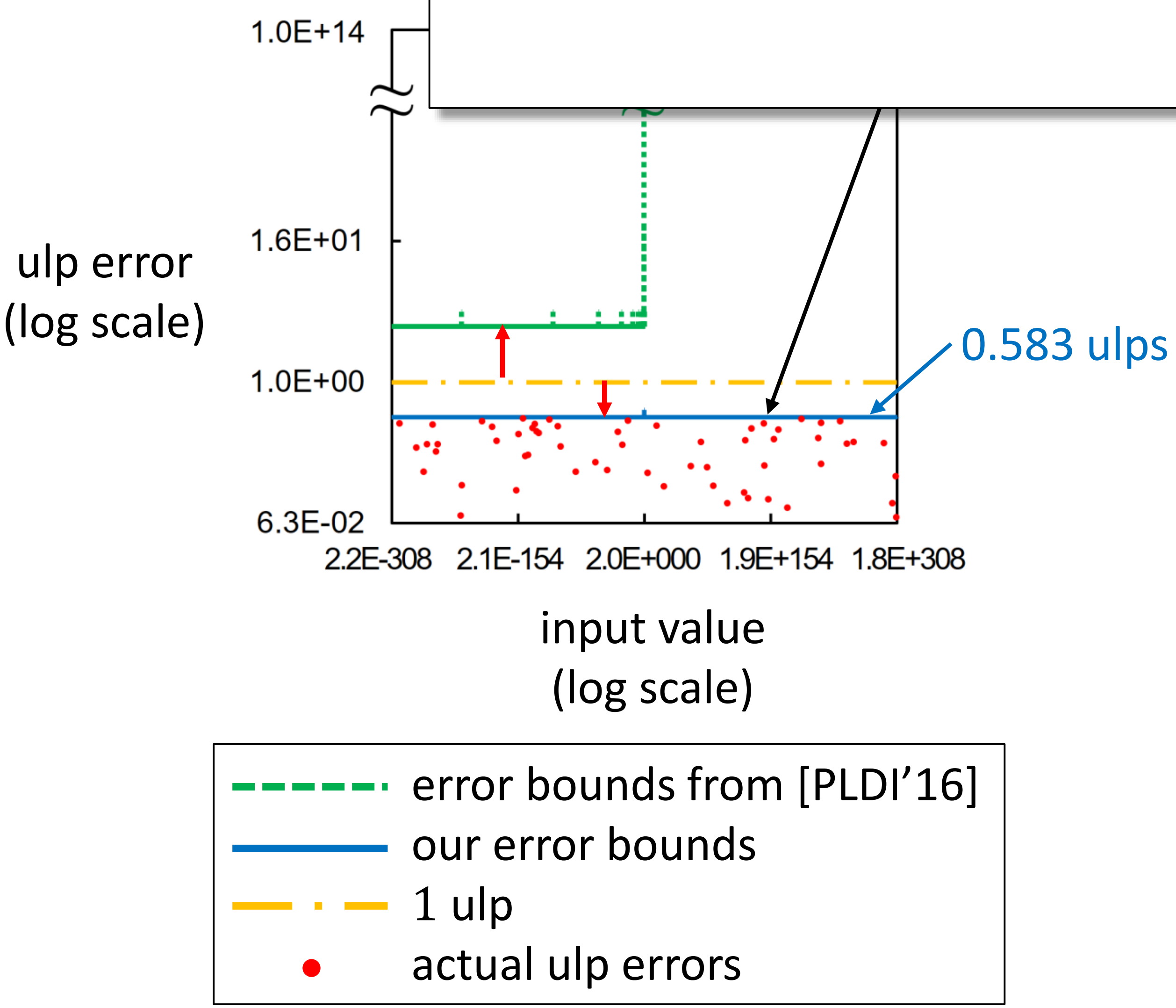

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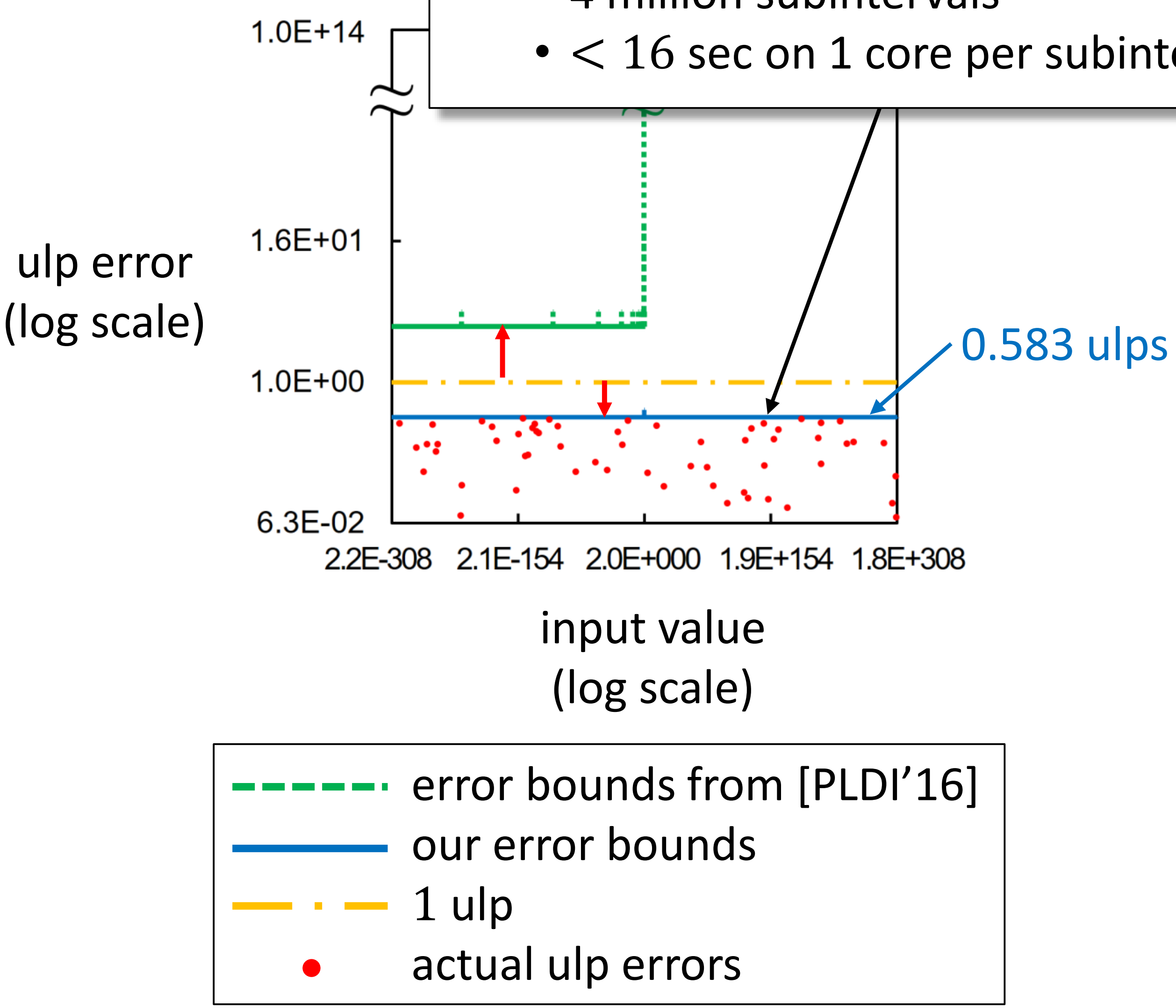
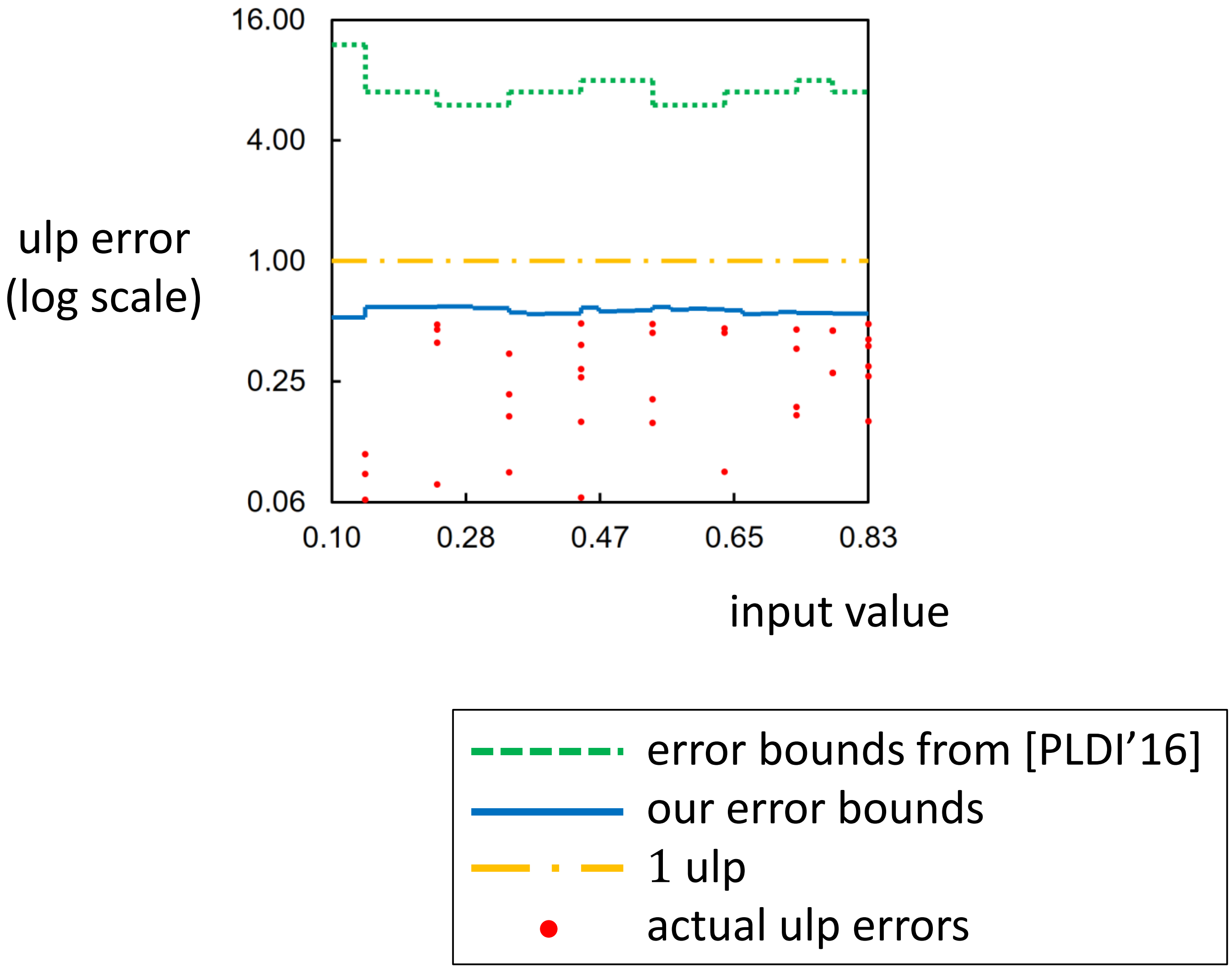
- Highly parallelizable:
  - 4 million subintervals
  - < 16 sec on 1 core per subinterval



Results: tan on $\left[\frac{13}{128}, \frac{17\pi}{64}\right), \left[\frac{17\pi}{64}, \frac{\pi}{2}\right)$



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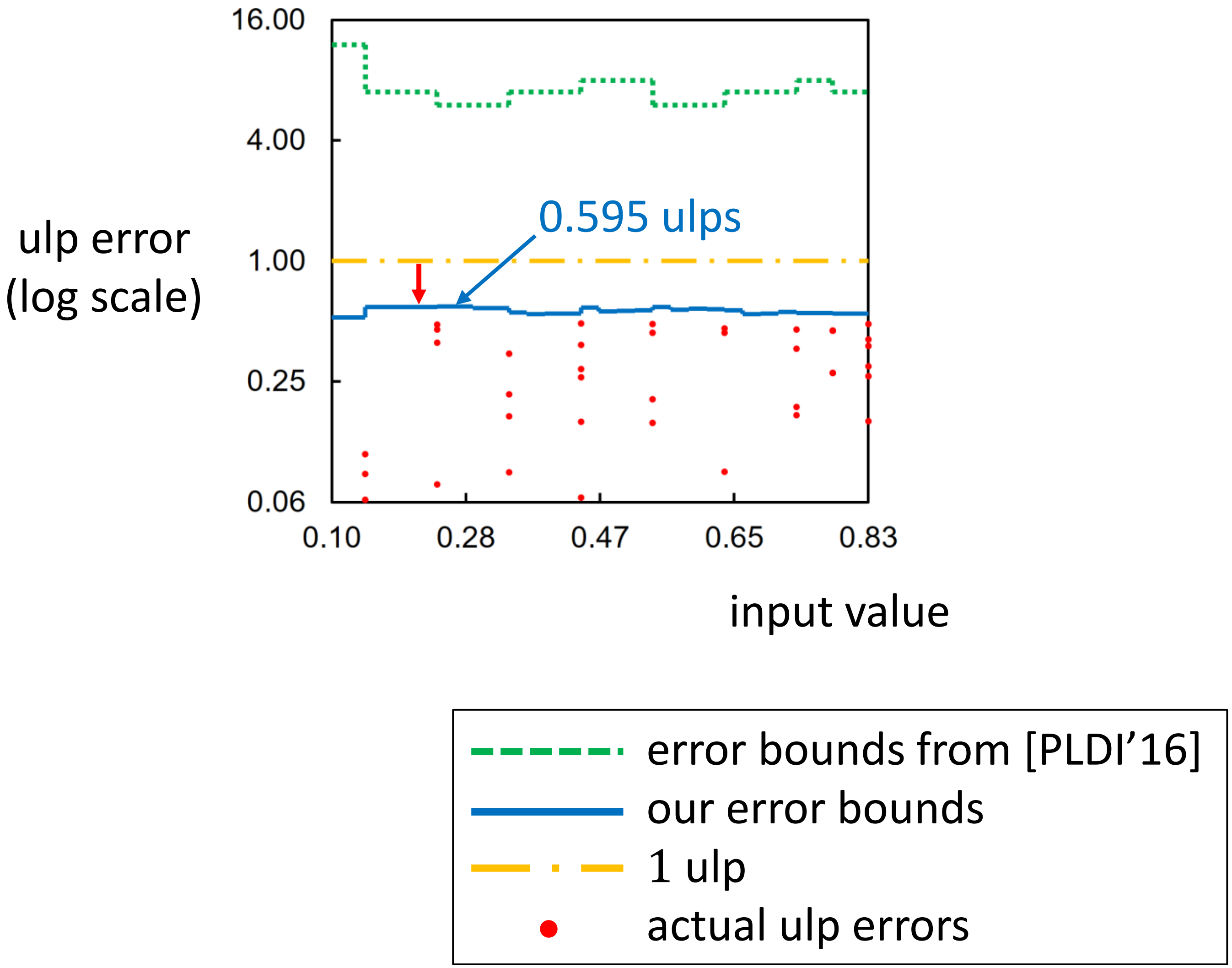

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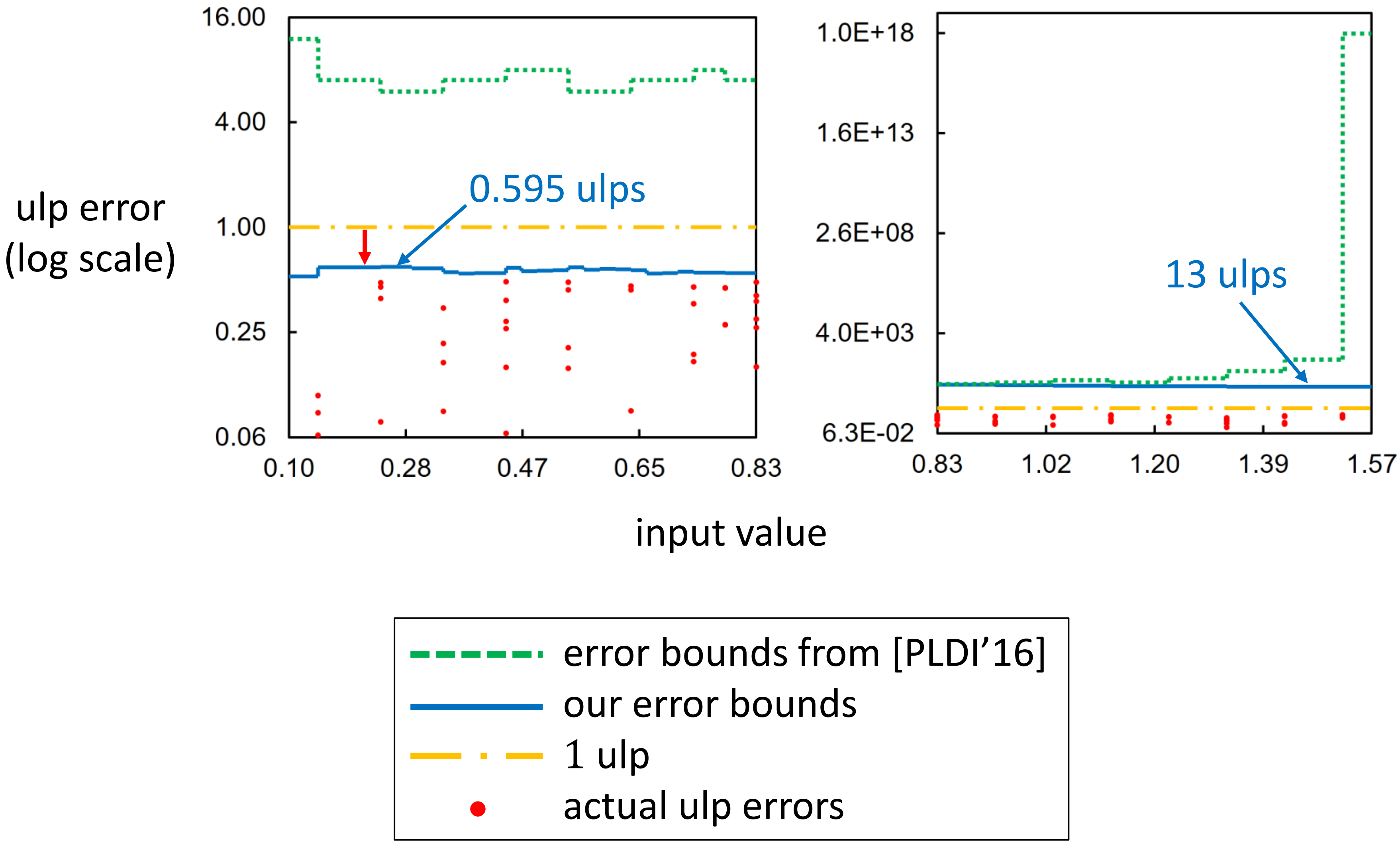

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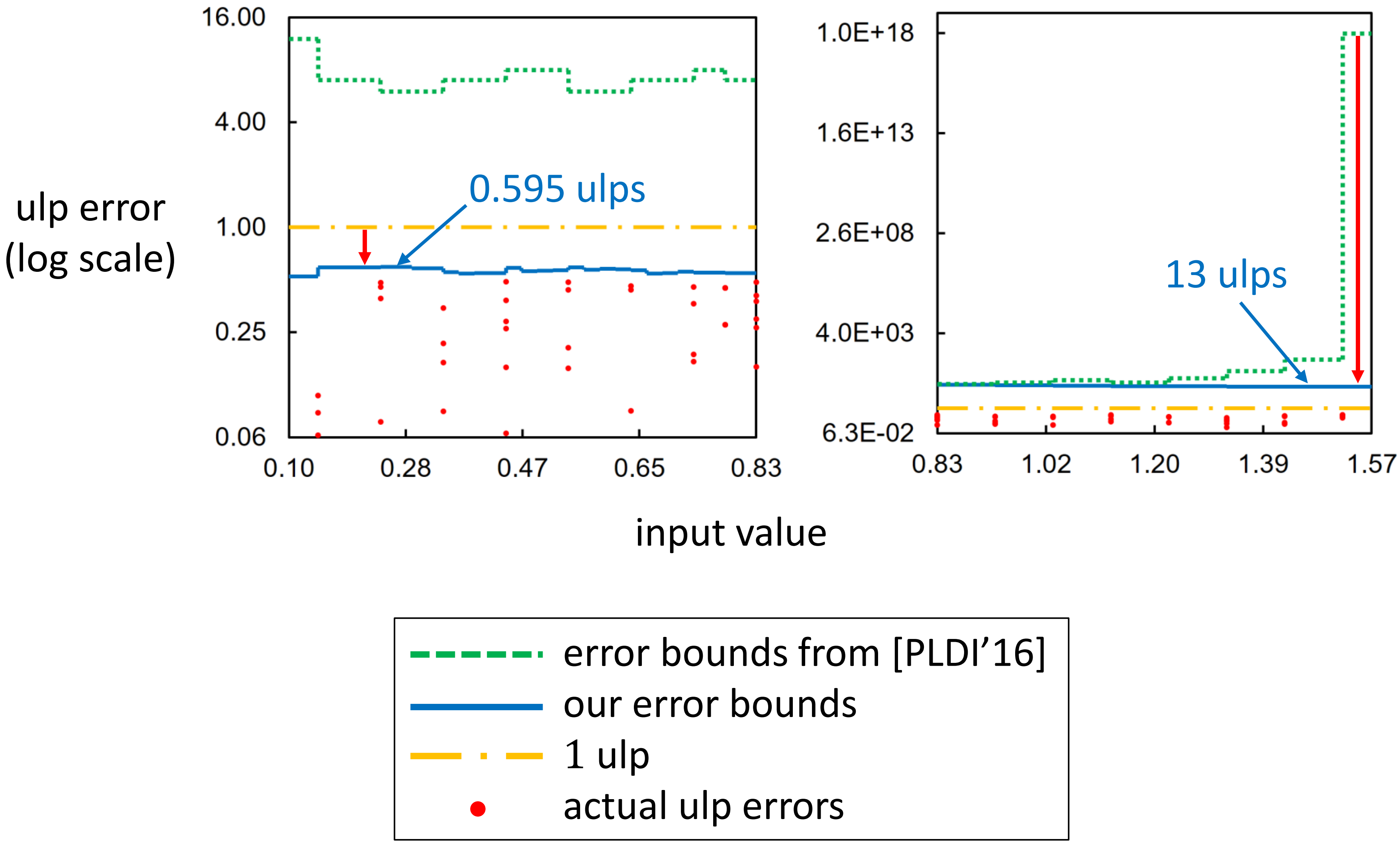

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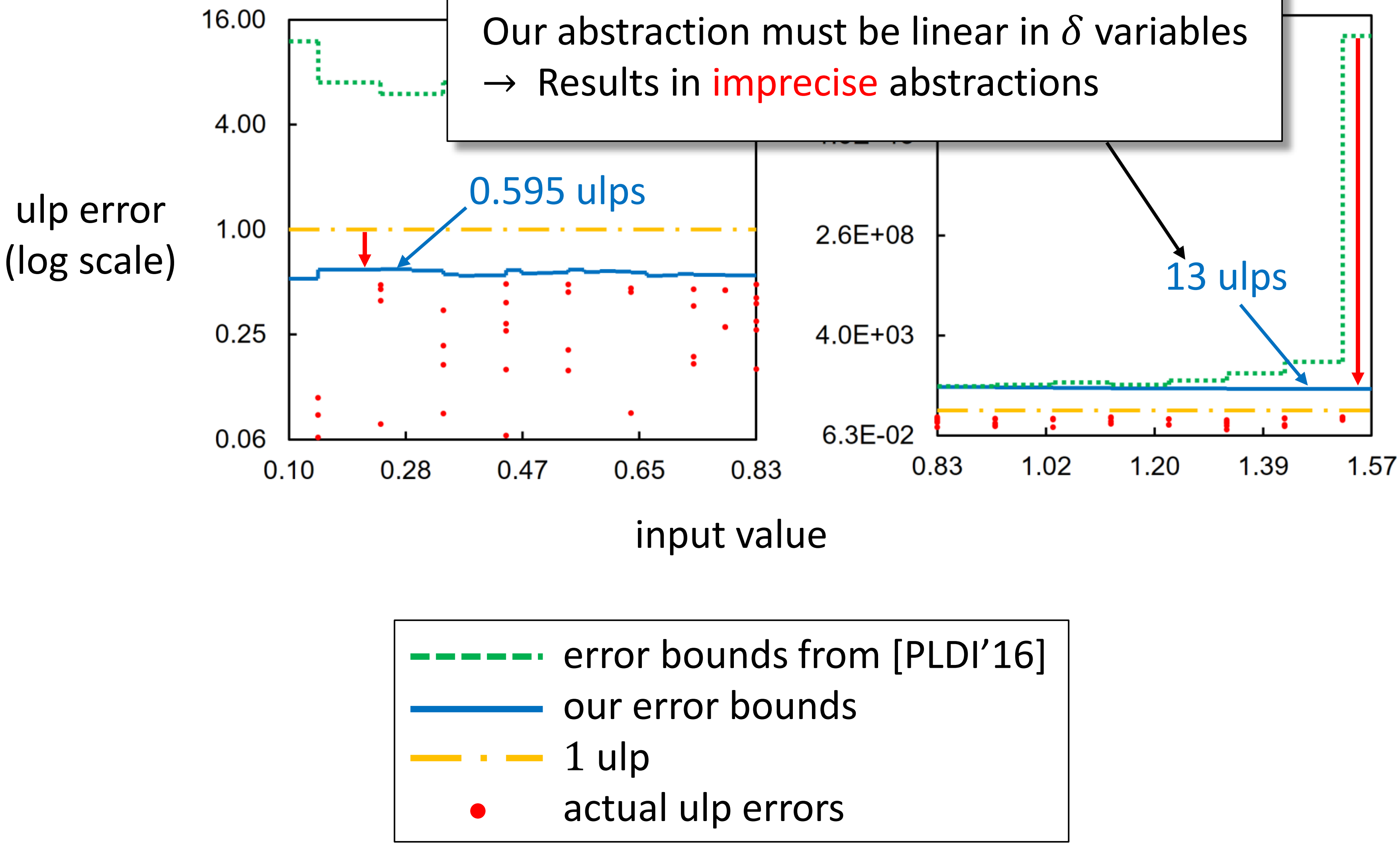
Our abstraction must be linear in δ variables
→ Results in imprecise abstractions



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