Reasoning about Floating Point in Real-World Systems

Wonyeol Lee (Stanford CS)

PhD Oral Exam (05/15/2023)

“Continuous” Computations

Continuous values

$$6,\ 2.5,\ \frac{3}{7},\ \sqrt{2},\ 0.9\pi,\ \ldots$$

Operations on them

$$6+2.5,\ \frac{3}{7}\times\sqrt{2},\ \cos(0.9\pi),\ \ldots$$

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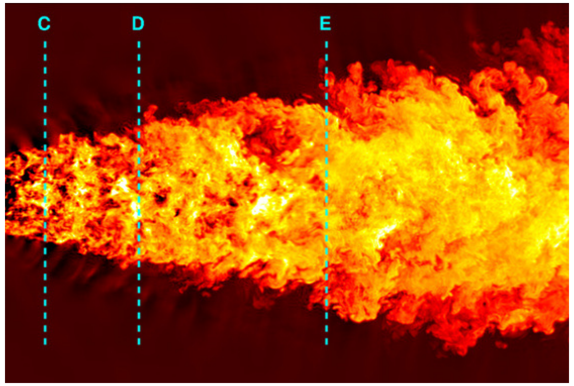
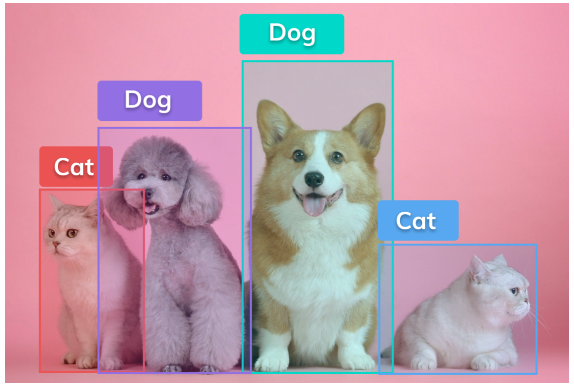


Scientific computing



Machine learning

Computer graphics

…

Theory and Practice

	Continuous values	Operations on them
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Theory and Practice

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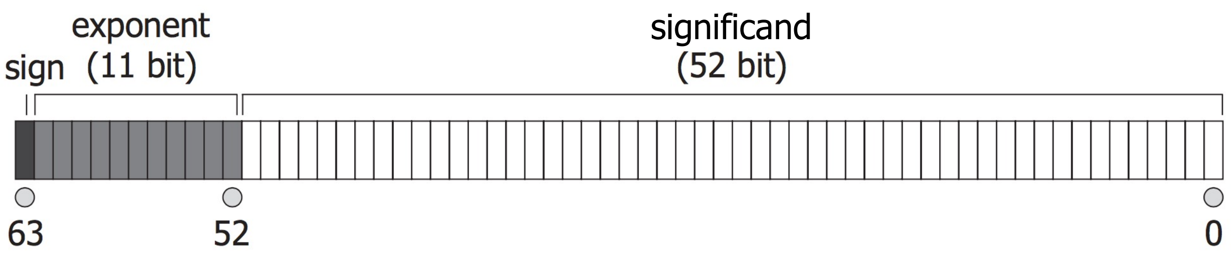
Practice	Floating-point numbers ($\mathbb{F}$)	Floating-point operations ($+_{\mathbb{F}}, \times_{\mathbb{F}}, \ldots$)

e.g., 64-bit double-precision floats



$$\underbrace{1}\ \underbrace{01111111111}\ \underbrace{1100\cdots00}_{(2)}$$

$$= (-1)^{1} \cdot 2^{1023-1023} \cdot 1.110\cdots00_{(2)}$$

Discrepancy

uncountably many　　　　exact

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- Discrepancy between the theory and practice of "continuous" computations.
- Can we better understand/characterize this discrepancy arising in real-world systems?

My Work

Programs that implement `math.h`.	Correctness: [PLDI'16], [POPL'18].
Programs that train ML models.	Acceleration: [Submitted].
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done during my 3-year leave from Stanford

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`math.h` Implementations

$$\log x$$

mathematical specification $\boldsymbol{f}$

$\boldsymbol{X} = (0, 2^{1024})$

input interval $\boldsymbol{X}$

```
<log>
...
subsd  %xmm5, %xmm1
mulpd  %xmm0, %xmm5
...
psllq  $0xc,  %xmm1
pshufd $0xe4, %xmm5, %xmm6
...
```

`math.h` implementation $\boldsymbol{P}$

`math.h` Implementations

infinite # bits

$\log x$

mathematical specification $\boldsymbol{f}$

∃ precision loss

$\boldsymbol{X} = (0, 2^{1024})$

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`math.h` implementation $\boldsymbol{P}$

Problem

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- Can we find a tight bound on the maximum precision loss automatically?

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`math.h` implementation $\boldsymbol{P}$

- Can we find a tight bound on the maximum precision loss automatically?
- Goal: Find a small $\Theta > 0$ in an automatic way such that

$$\mathrm{err}\big(\boldsymbol{f}(x), \boldsymbol{P}(x)\big) \le \Theta \quad \text{for all } x \in \boldsymbol{X}.$$

Two Challenges

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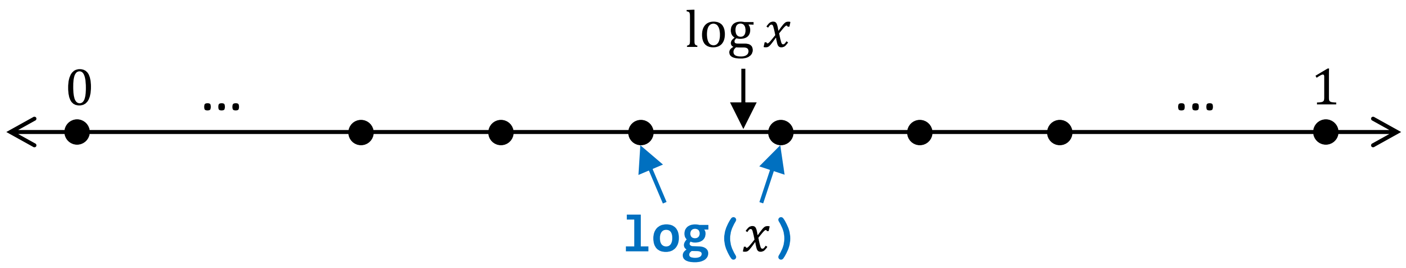
(2) $\boldsymbol{P}$ is often claimed to have a very small precision loss. [POPL'18]

e.g., < 1 ulp

Two Challenges

- `log` has precision loss of < 1 ulp $\iff$ for any $x \in \boldsymbol{X}$,



- 0.5 ulp is the best we can achieve (by definition).

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Two Challenges

Prior work on the problem [FM’15, POPL’14, FMICS’09, PLDI’03, FMCAD’00, …]:

- requires considerable manual efforts; or
- cannot handle general mixed codes and prove small error bounds.

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- Example: Given n (in `int`), compute 2^n (in `double`).
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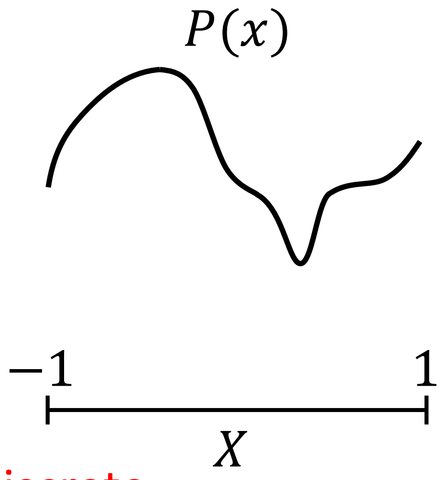
  - Better solution: `(n + 1023) << 52`. Fast, correct for $n \in [-1022, 1023]$.

- Such operations are often used in highly optimized implementations of `math.h`.

It is difficult to reason about such "mixed codes", which intermix bit-level and floating-point operations.

Bit-Level Operations

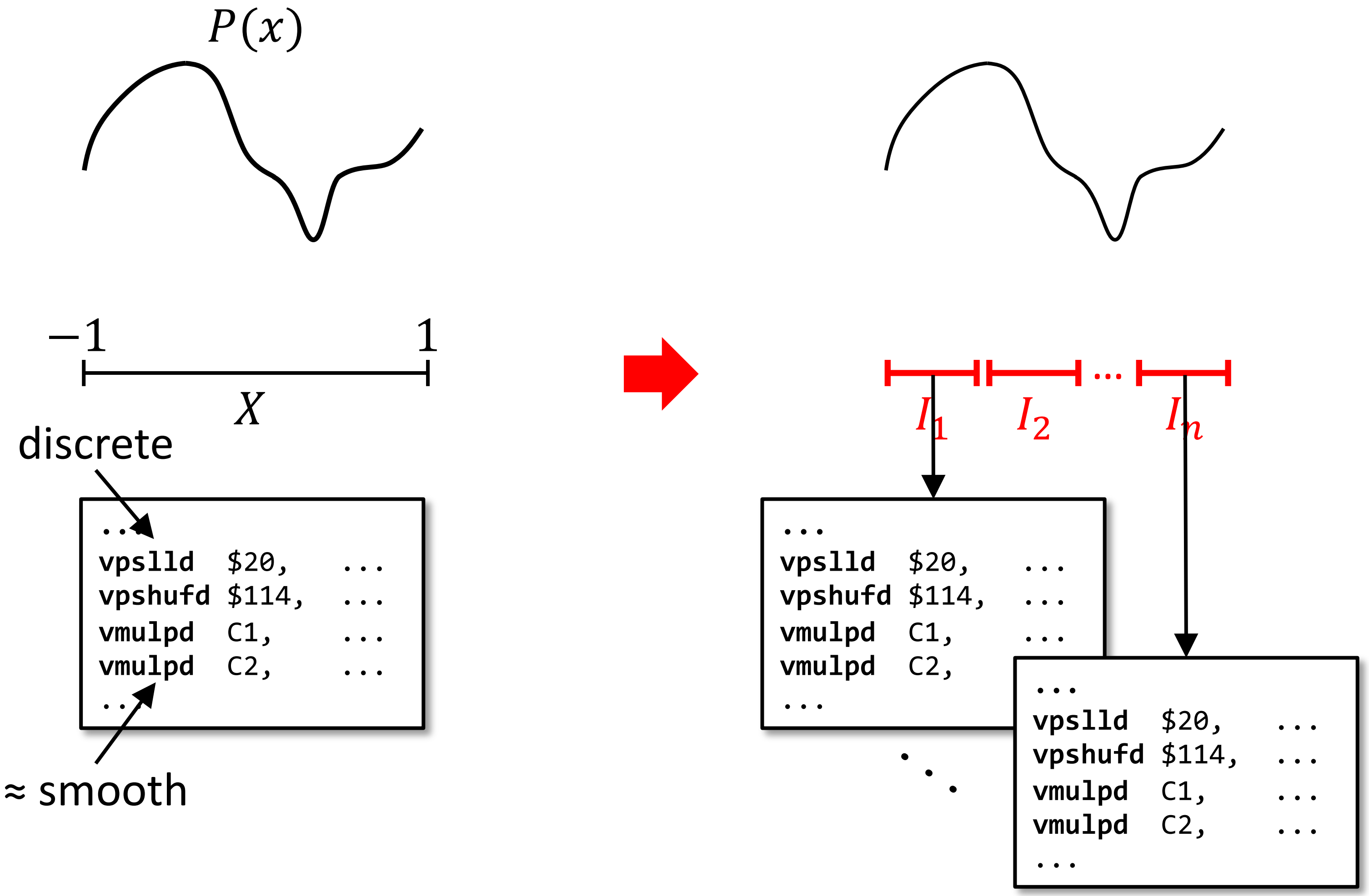


discrete

```
...
vpslld  $20,    ...
vpshufd $114,   ...
vmulpd  C1,     ...
vmulpd  C2,     ...
..
```

≈ smooth

Our Approach



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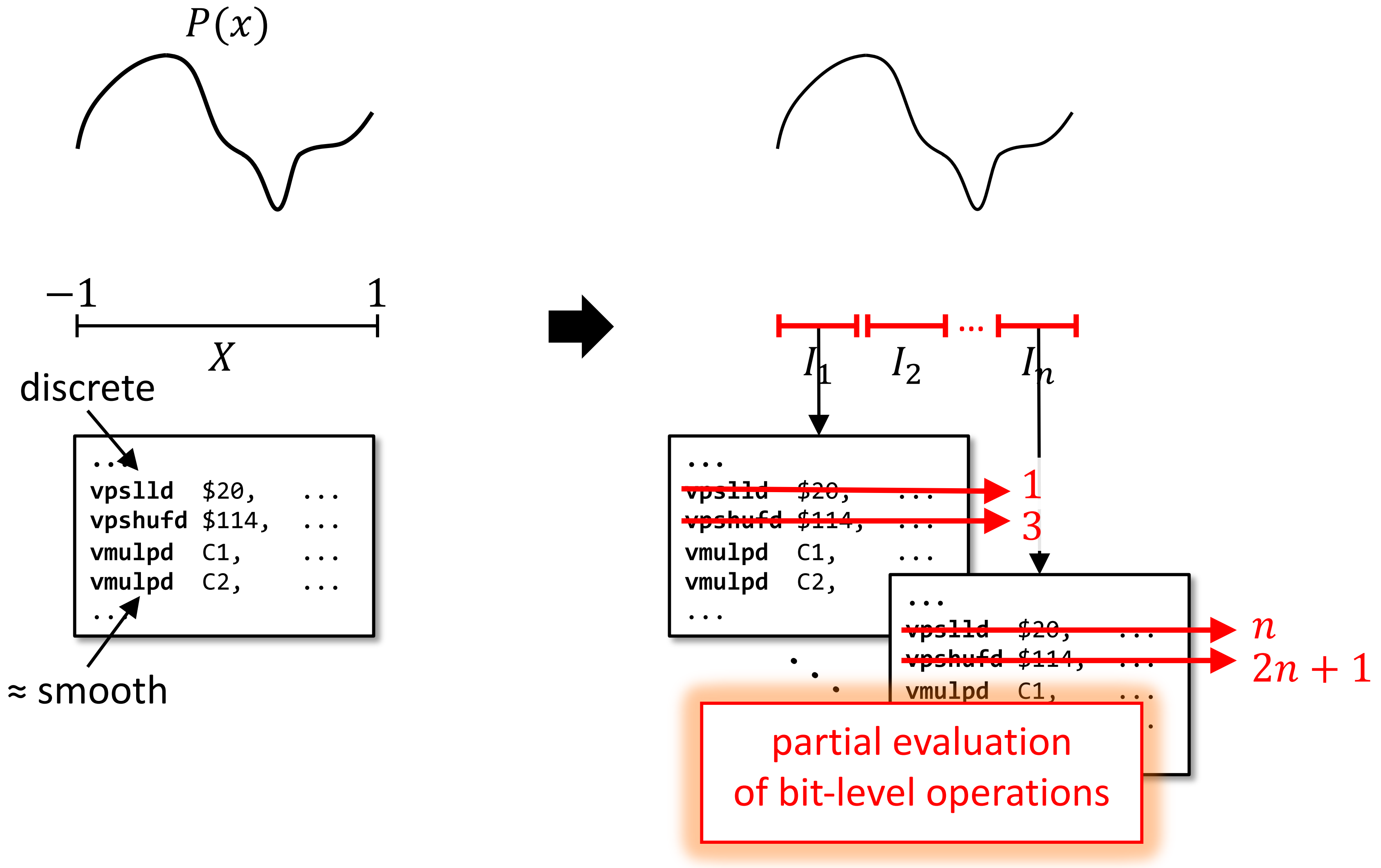

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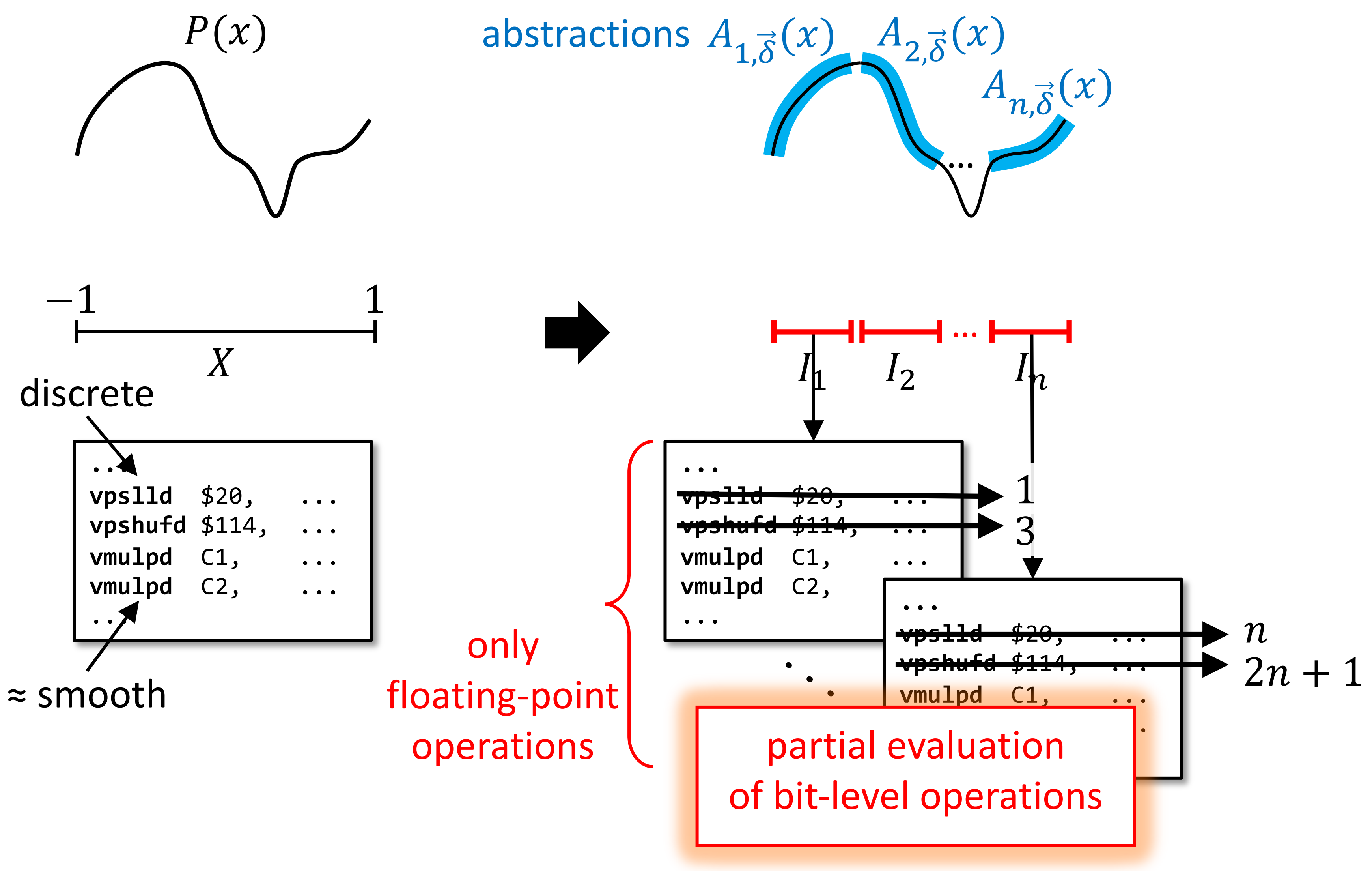

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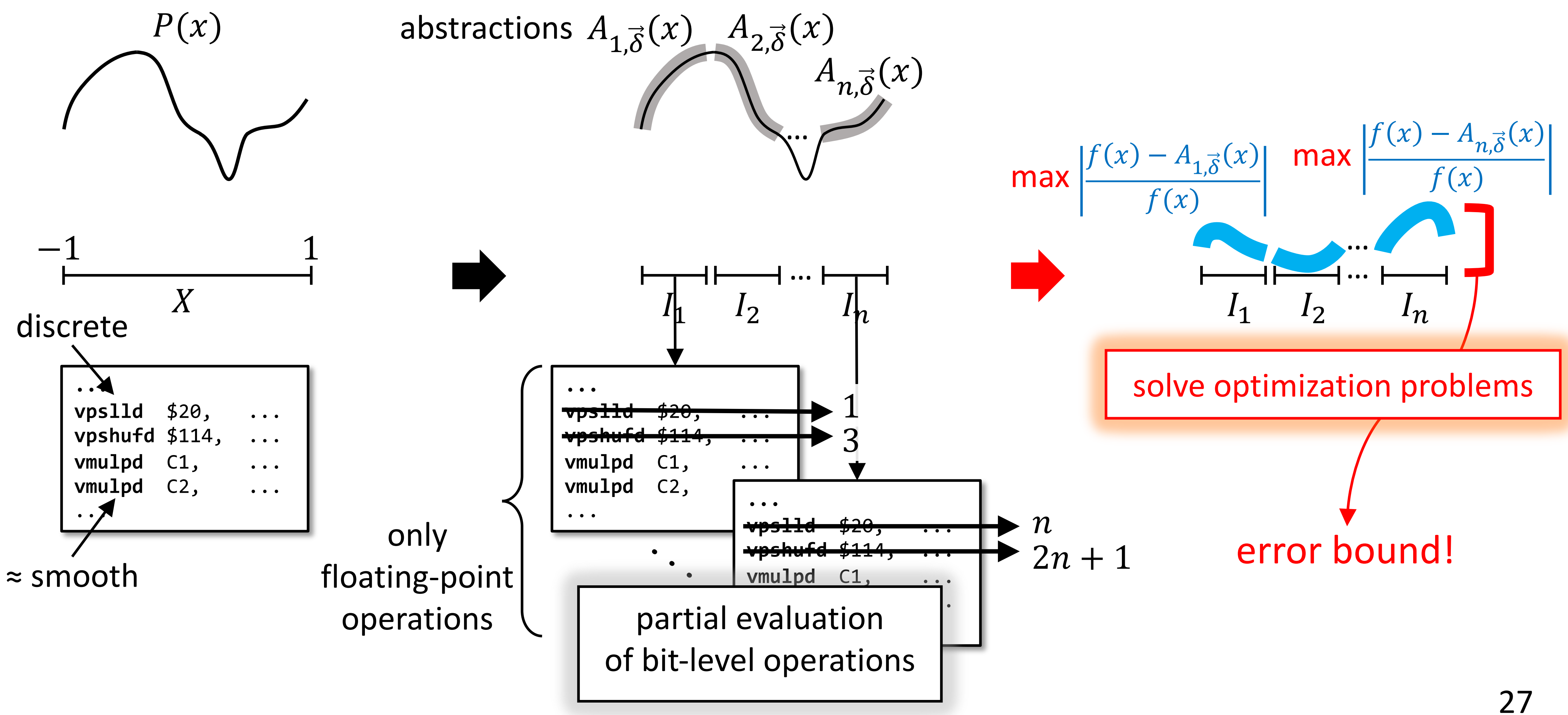
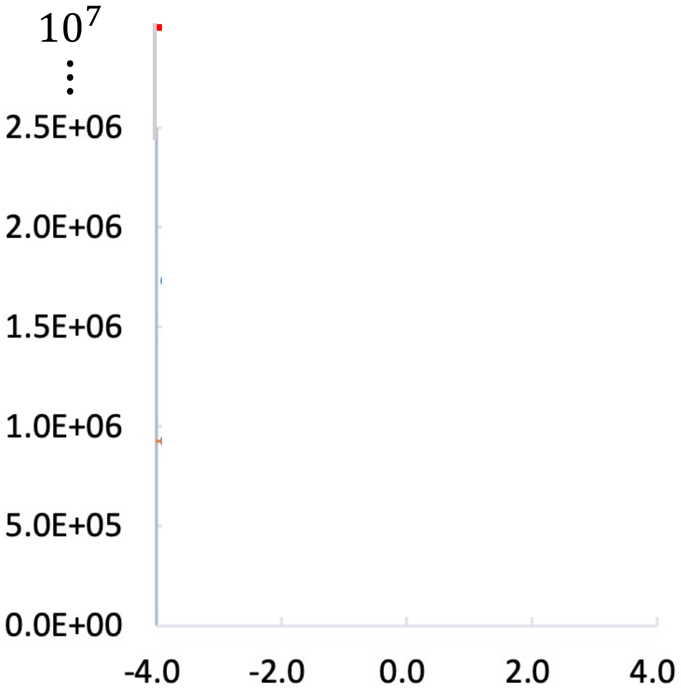
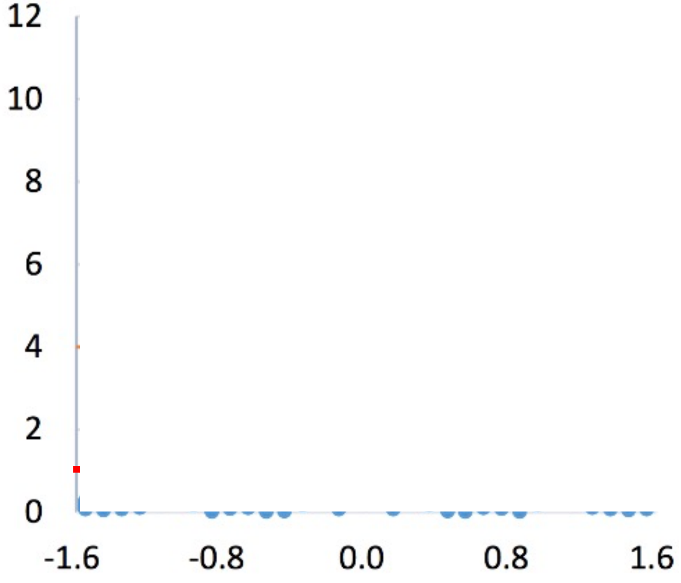


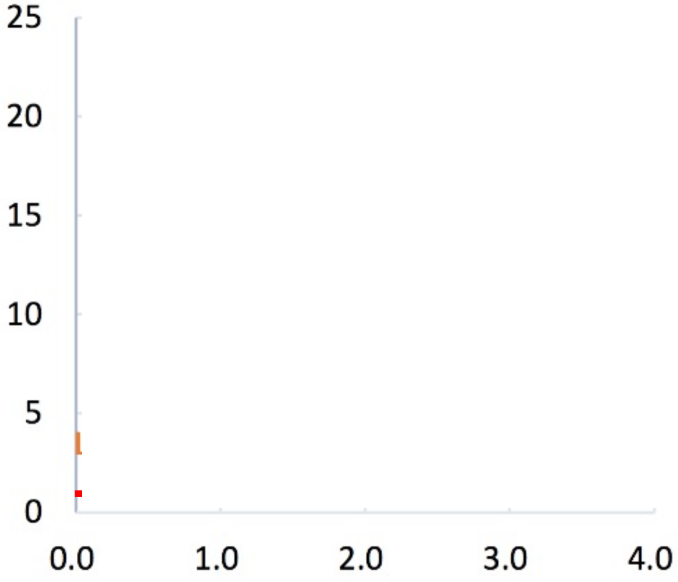
Evaluation Results

Optimized implementation	Intel's implementation of `math.h`	
$\mathtt{exp}_{\mathtt{opt}}$	`sin`	`log`

error (ulp)







input

Evaluation Results

Our error bounds [PLDI'16]
Claimed error bounds
Actual errors (between intervals)

Optimized implementation	Intel's implementation of `math.h`	
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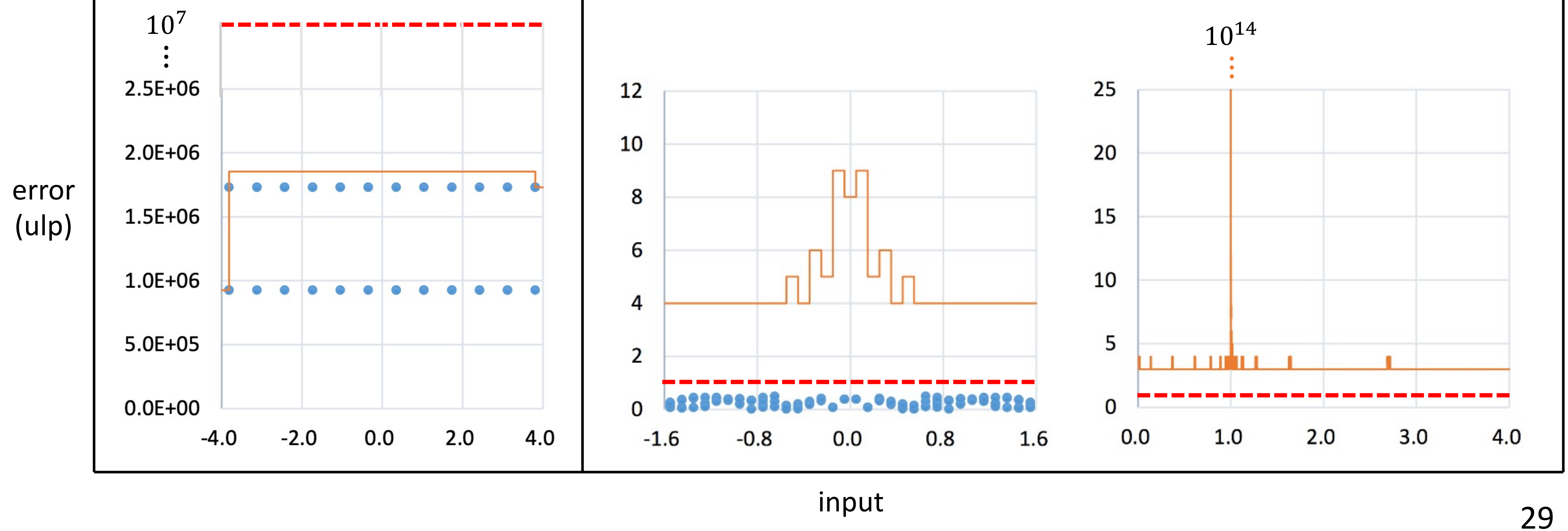

Summary of Contributions [PLDI'16]

- We propose the first systematic, automatic method for verifying mixed codes.
- Our method is based on abstraction, analytic optimization, and testing.

  Key: Split the input range into sub-intervals so that bit-level op's can be partially evaluated.
- We apply our method to real-world binaries for `math.h` and prove their formal error bonds.

Two Challenges

$\log x$ ⟷

mathematical specification $\boldsymbol{f}$

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Exactness Properties

- Floating-point operations are often inexact.

$$a \times_{\mathbb{F}} 2^n \neq a \times 2^n$$

$$a -_{\mathbb{F}} b \neq a - b$$

Exactness Properties

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$$a \times_{\mathbb{F}} 2^n = a \times 2^n \quad \text{if } |a \times 2^n| \geq 2^{-1022}. \qquad \text{[Folklore]}$$

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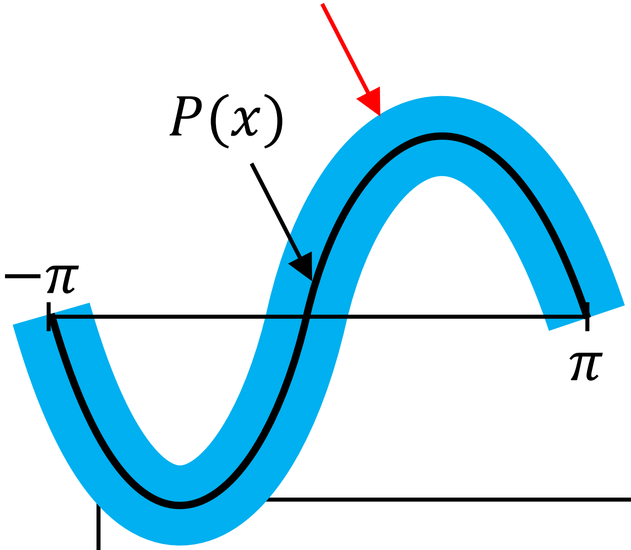
$$[\cdots] \qquad \text{if} \qquad [\cdots]$$

- Such properties are implicitly used in highly accurate implementations of `math.h`.

Standard error analysis techniques ignore these exactness properties.

Loose Error Bounds

This sometimes results in
too overapproximate abstractions.



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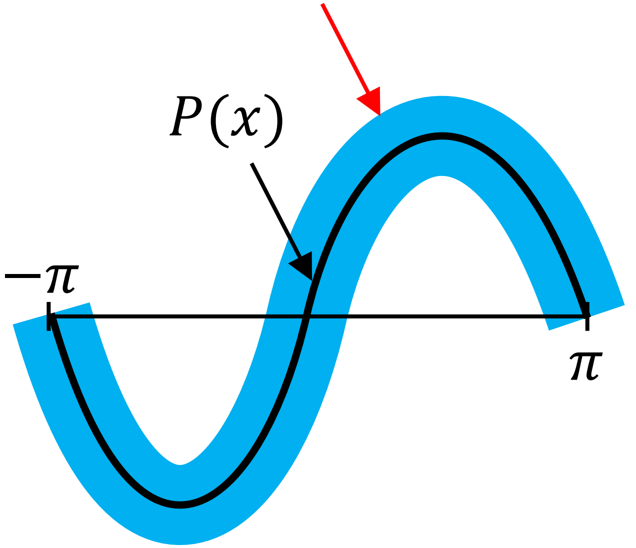
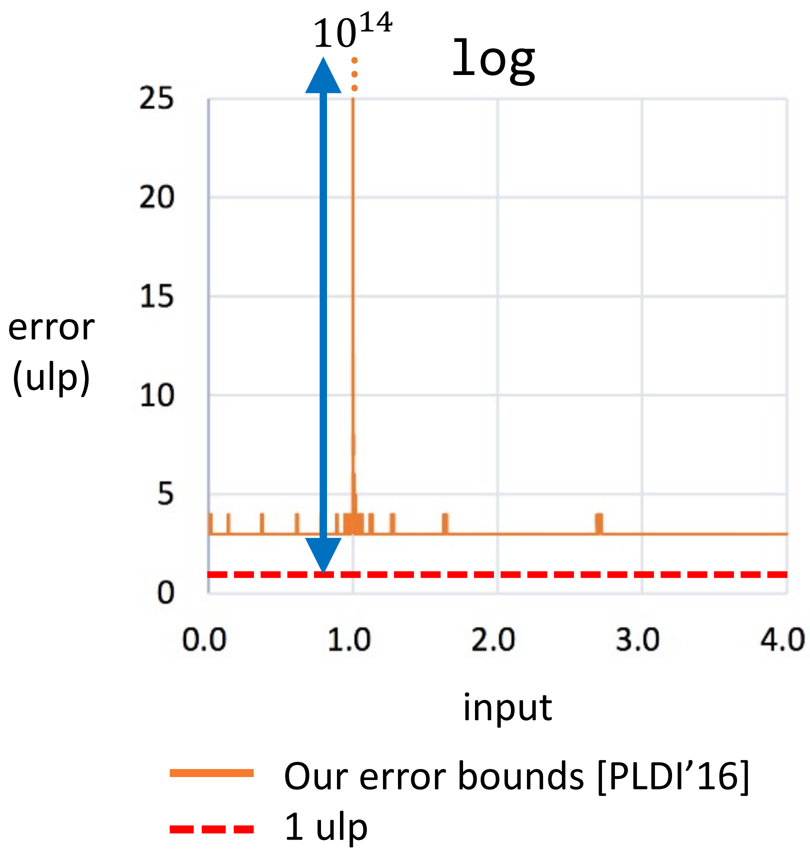


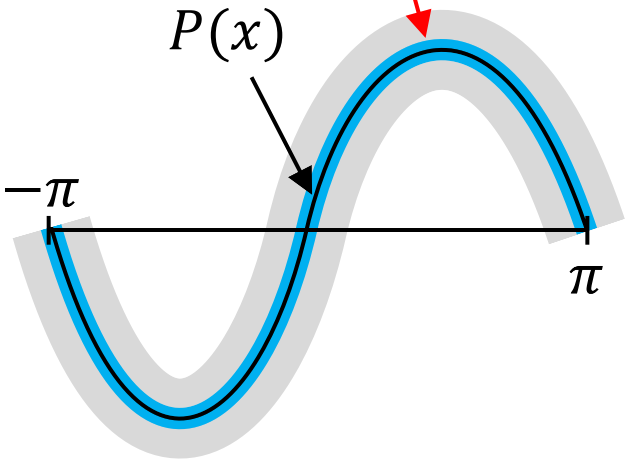
$\Rightarrow$

Intel's implementation of `math.h`



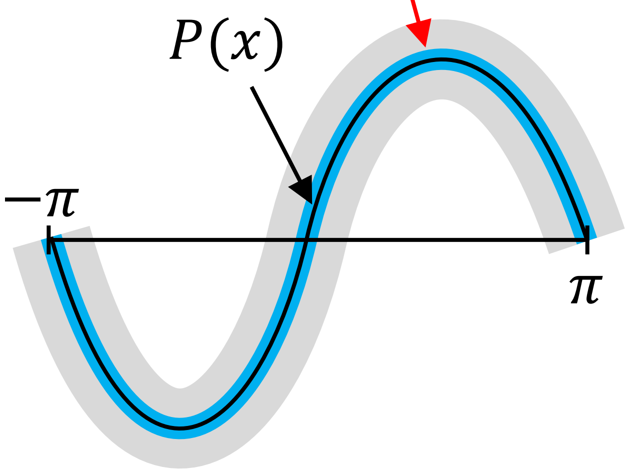
Our Approach

Construct tighter abstractions by automatically applying exactness results.



Our Approach

Construct tighter abstractions by automatically applying exactness results. ➡ Check preconditions of exactness results.



Example: Can we apply "$e(x) -_{\mathbb{F}} e'(x) = e(x) - e'(x)$"?

- Need: ${}^{1}/_{2}\, e(x) \leq e'(x) \leq 2e(x)$ for all $x \in I$.

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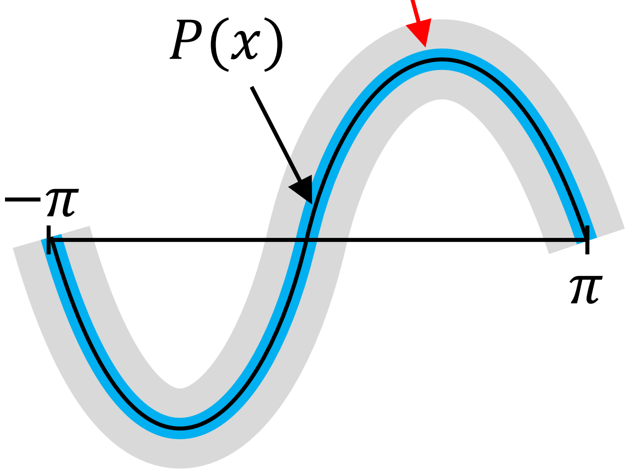

Check preconditions of exactness results.

Solve optimization problems.

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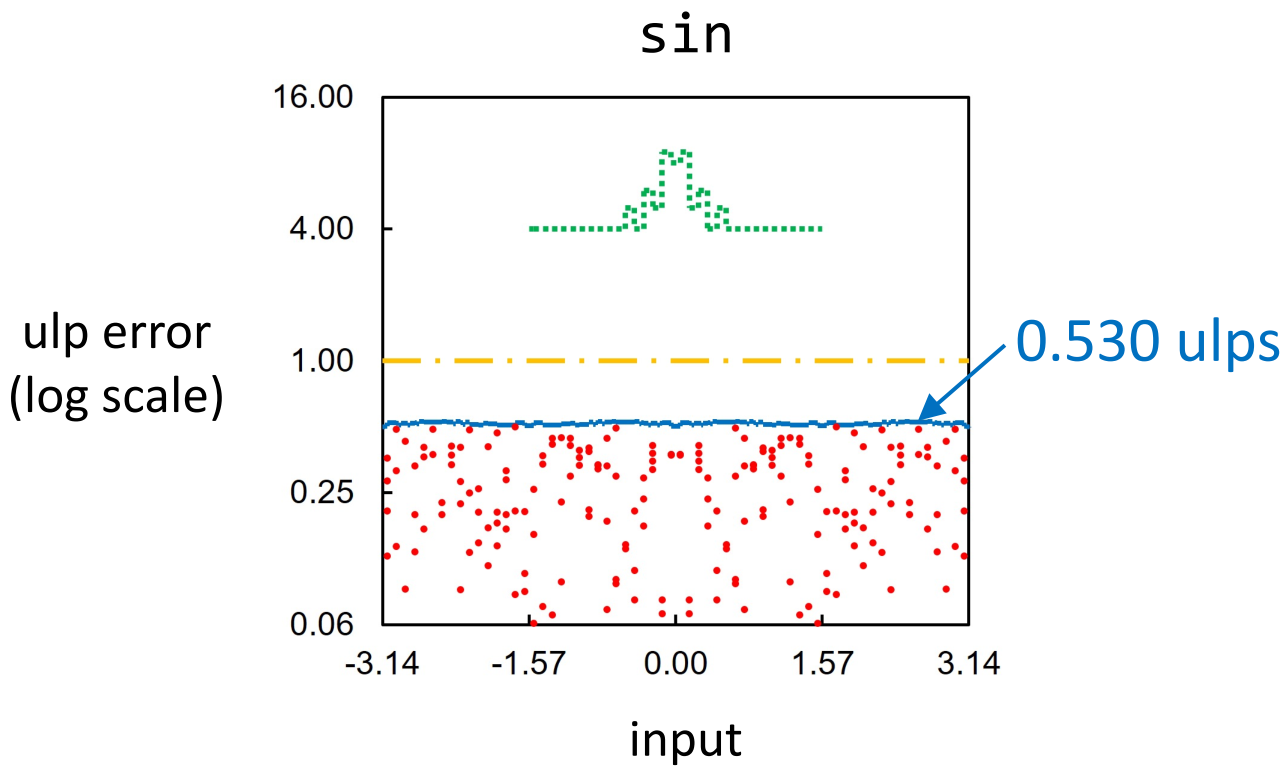
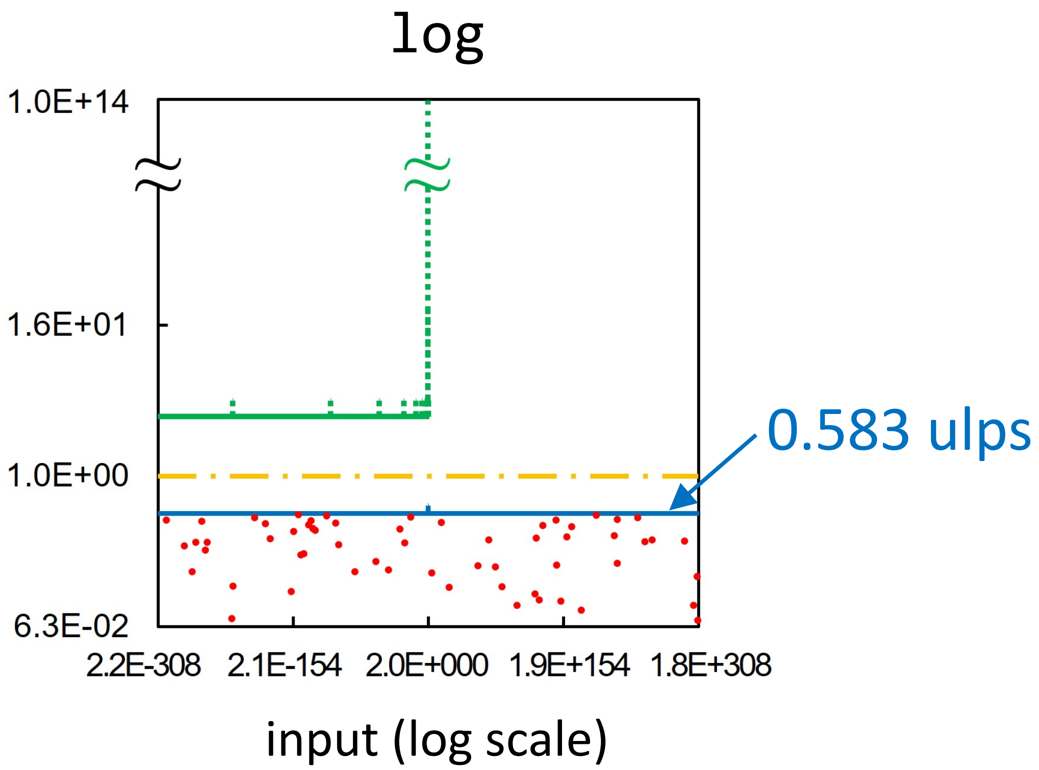
- Need: ${}^1\!/_2\, e(x) \le e'(x) \le 2e(x)$ for all $x \in I$.
- Check: $\min_{x \in I}\left(e'(x) - {}^1\!/_2\, e(x)\right) \ge 0$
  $\land \max_{x \in I}\left(e'(x) - 2e(x)\right) \le 0.$

Evaluation Results

error bounds from [PLDI'16]
error bounds from [POPL'18]
1 ulp
actual ulp errors

Intel's implementation of `math.h`





Summary of Contributions [POPL'18]

- We propose the first automatic method for verifying `math.h` implementations.
- Our method is based on a reduction of this verification problem to optimization problems.
  Key: Apply exactness results, after checking their preconditions by solving opt'n problems.
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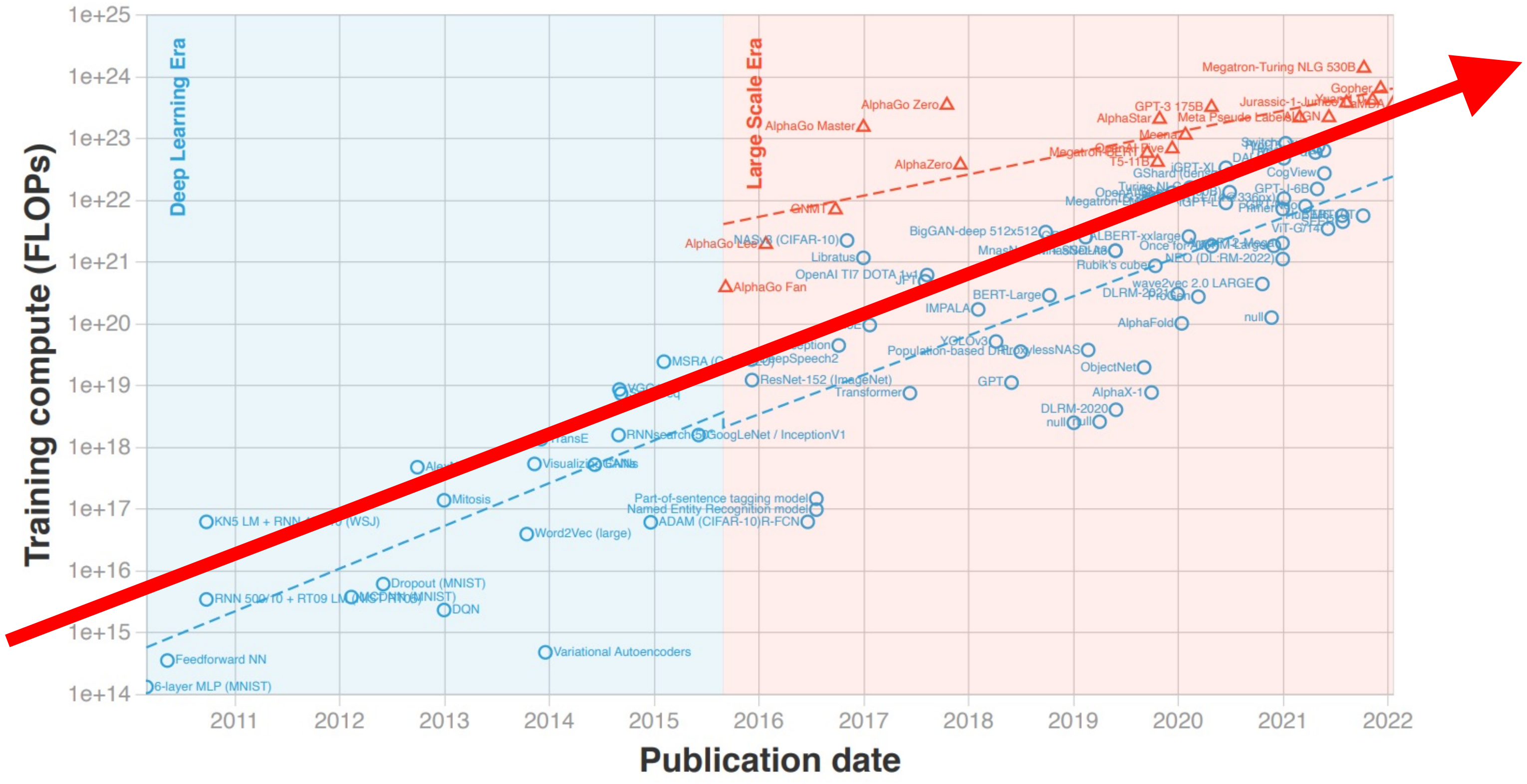
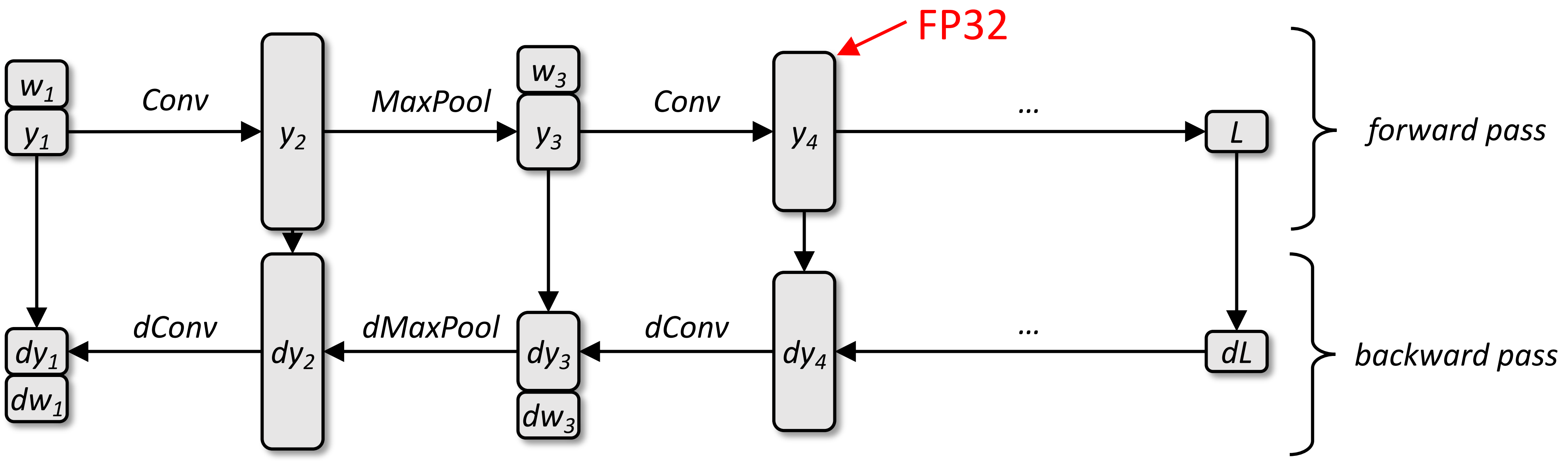
Training in Machine Learning



How to accelerate training computation while maintaining training quality?

Source: “Compute trends across three eras of machine learning”, IJCNN’22.

Low-Precision Training



- Standard training: Use FP32 to represent tensors.

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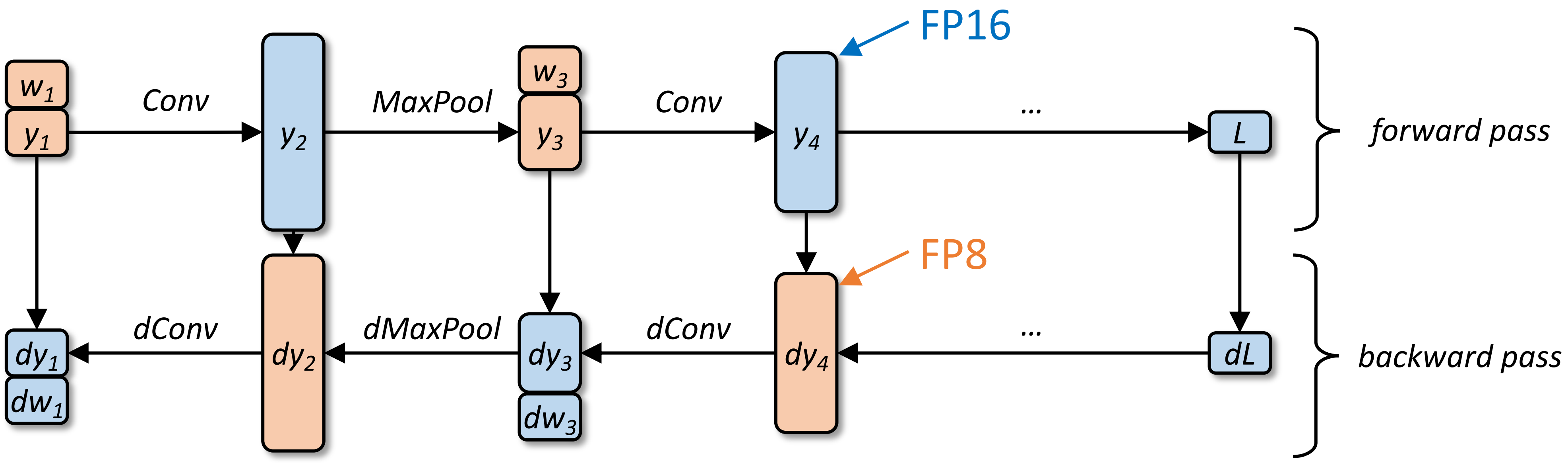

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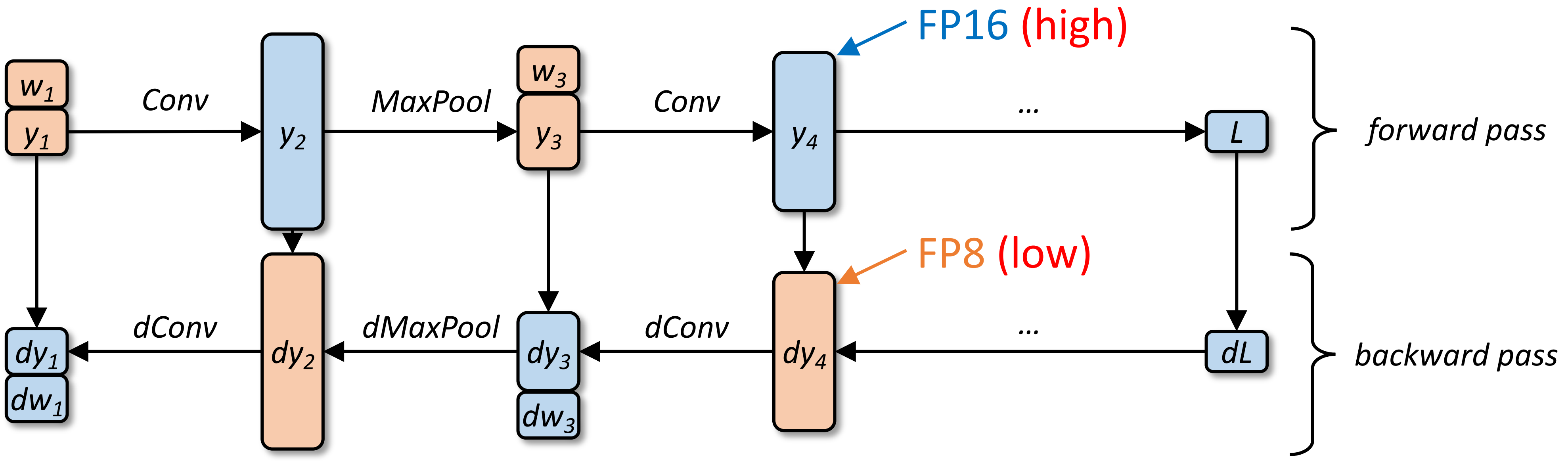

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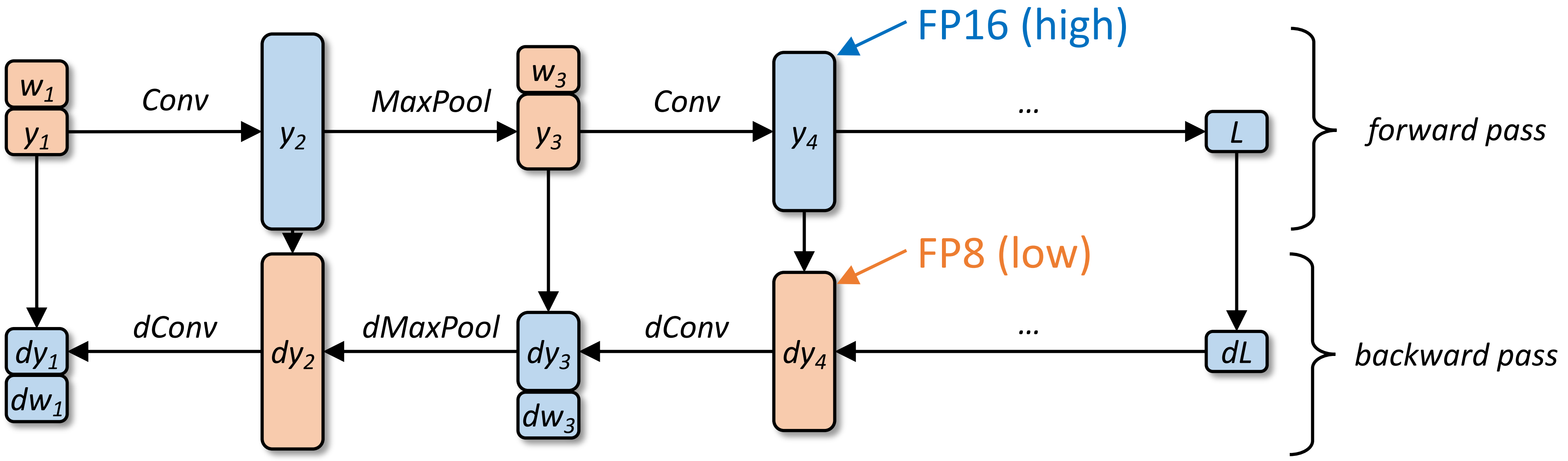
Low-Precision Training

$\pi_{\text{op}} =$



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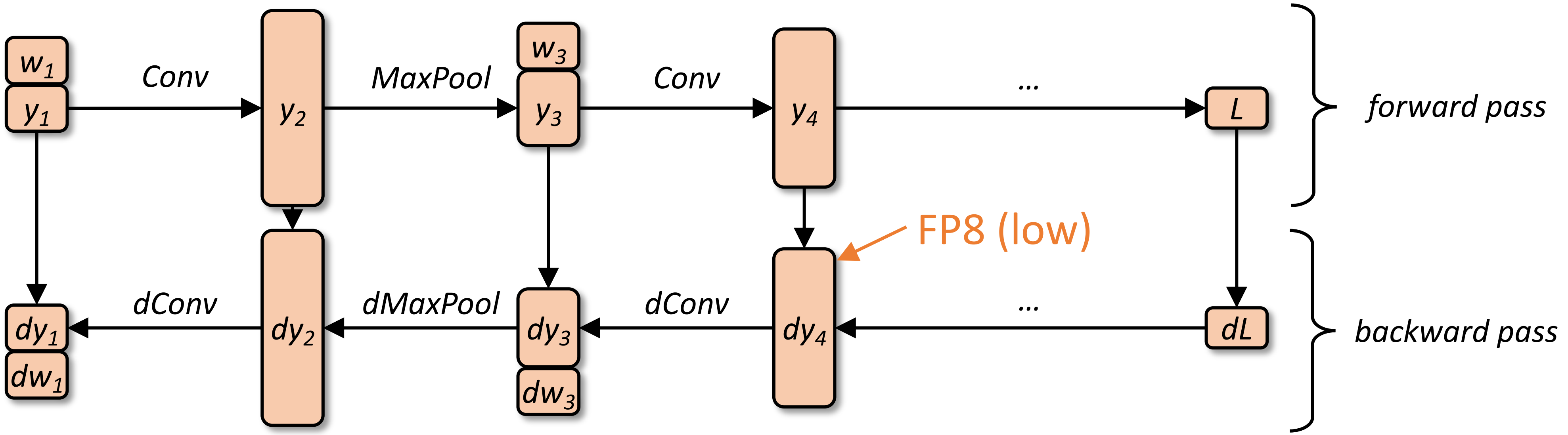
Low-Precision Training

$\pi_{\text{unif}} =$



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  - Example: operator-based assignment π_{op}, uniform assignment π_{unif}.

Limitations

most often just one

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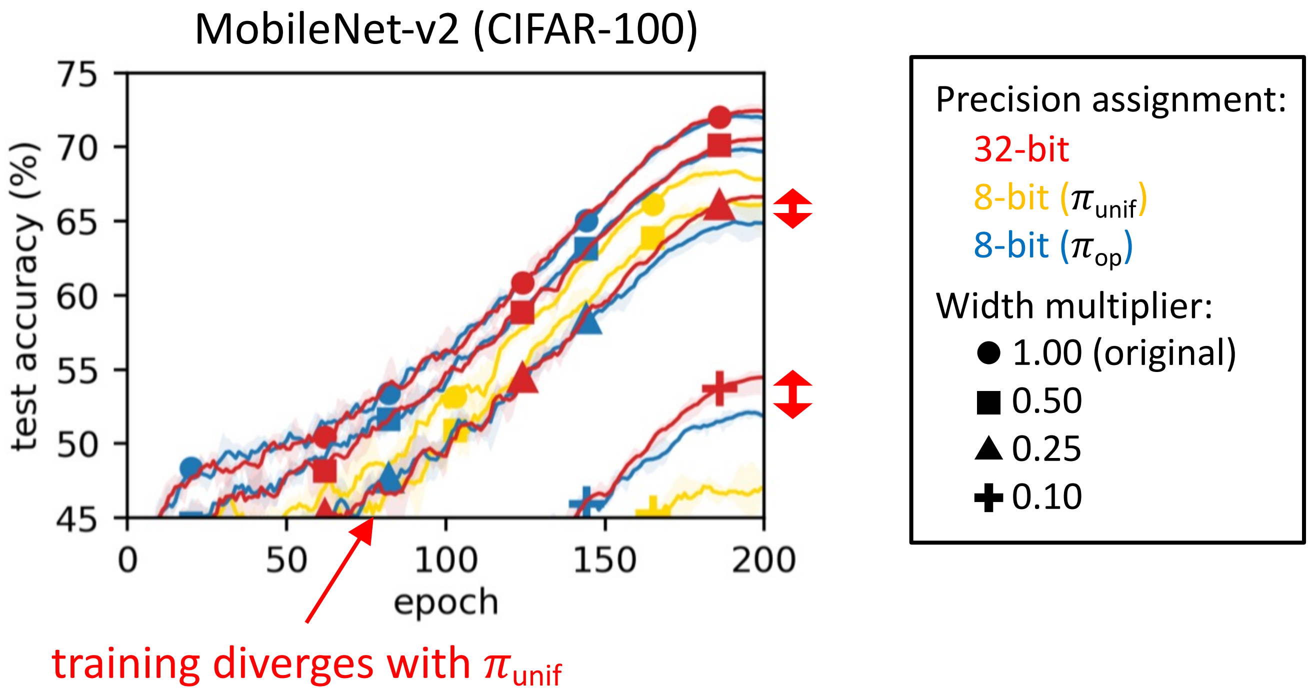

Memory-Accuracy Tradeoff Problem

- Input: a model M, FP formats $(FP_{\mathrm{hi}}, FP_{\mathrm{lo}})$, and a parameter $r \in [0,1]$.
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$\mathrm{acc}_M(\pi)$ is maximized subject to $\mathrm{ratio}_{\mathrm{lo}}(\pi) \geq r$.

accuracy of M when trained with π

"low-precision ratio" of π

$=$ ratio of FP_{lo}-tensors in π

$\approx$ reduction in memory/time for training

E.g., $\mathrm{ratio}_{\mathrm{lo}}(\pi_{\mathrm{hi}}) = 0$, $\mathrm{ratio}_{\mathrm{lo}}(\pi_{\mathrm{lo}}) = 1$.

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"low-precision ratio" of π
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- Find π that maximizes training accuracy under a memory/time constraint (given by r).

Challenges

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$$\text{acc}_M(\pi) \text{ is maximized} \qquad \text{subject to} \qquad \text{ratio}_{\text{lo}}(\pi) \geq r.$$

- Practically,
  - There is no known analytic method for predicting $\text{acc}_M(\pi)$.
  - There are exponentially many candidates for π.

Challenges

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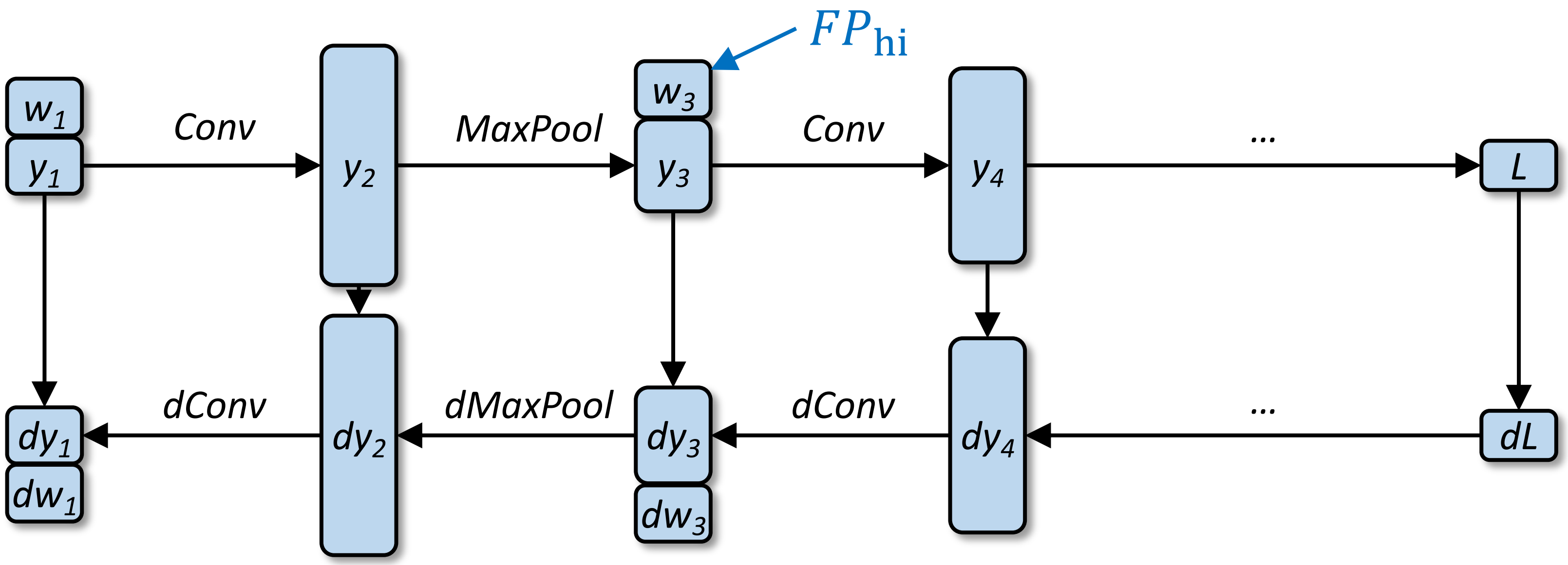
- Practically,
  - There is no known analytic method for predicting $\mathrm{acc}_M(\pi)$.
  - There are exponentially many candidates for π.
- Theoretically, we prove:

  <u>Theorem</u> The memory-accuracy tradeoff problem is NP-hard.

Our Method

- Input: a model M, FP formats $(FP_{\text{hi}}, FP_{\text{lo}})$, and a parameter $r \in [0,1]$.
- Our method for **the tradeoff problem:**
  - Initialize π to the all-FP_{hi} assignment.

$\pi =$



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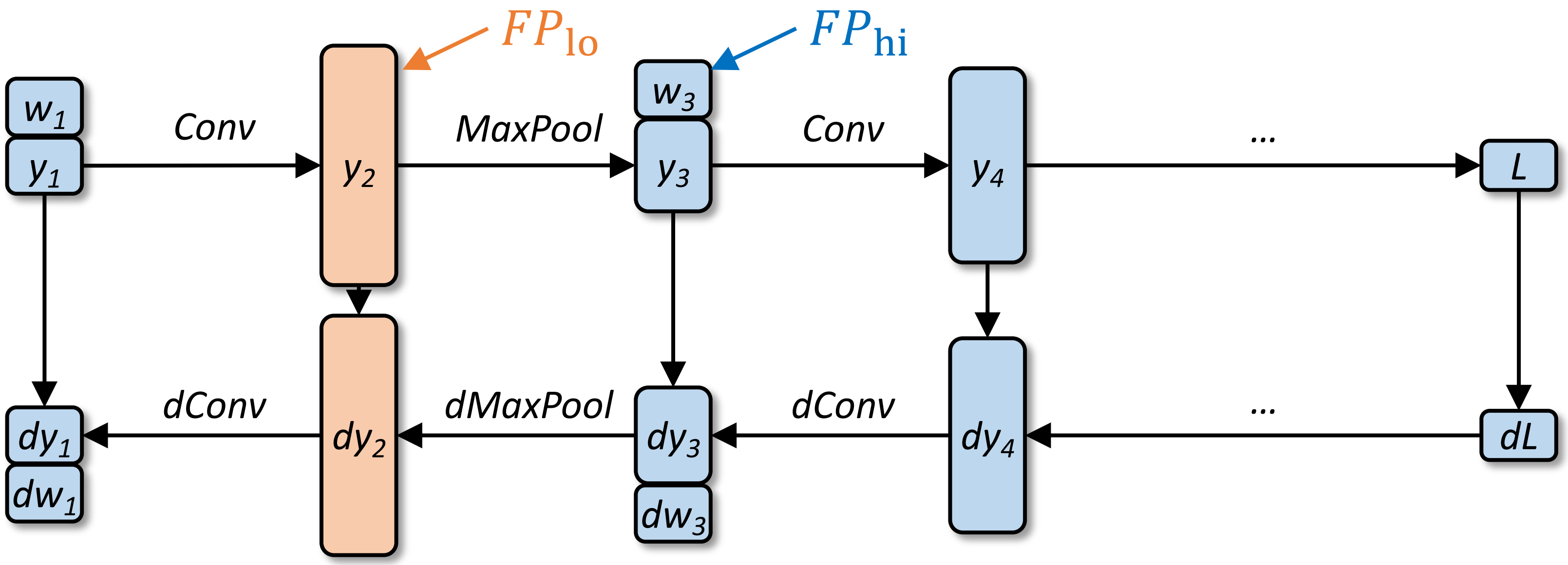

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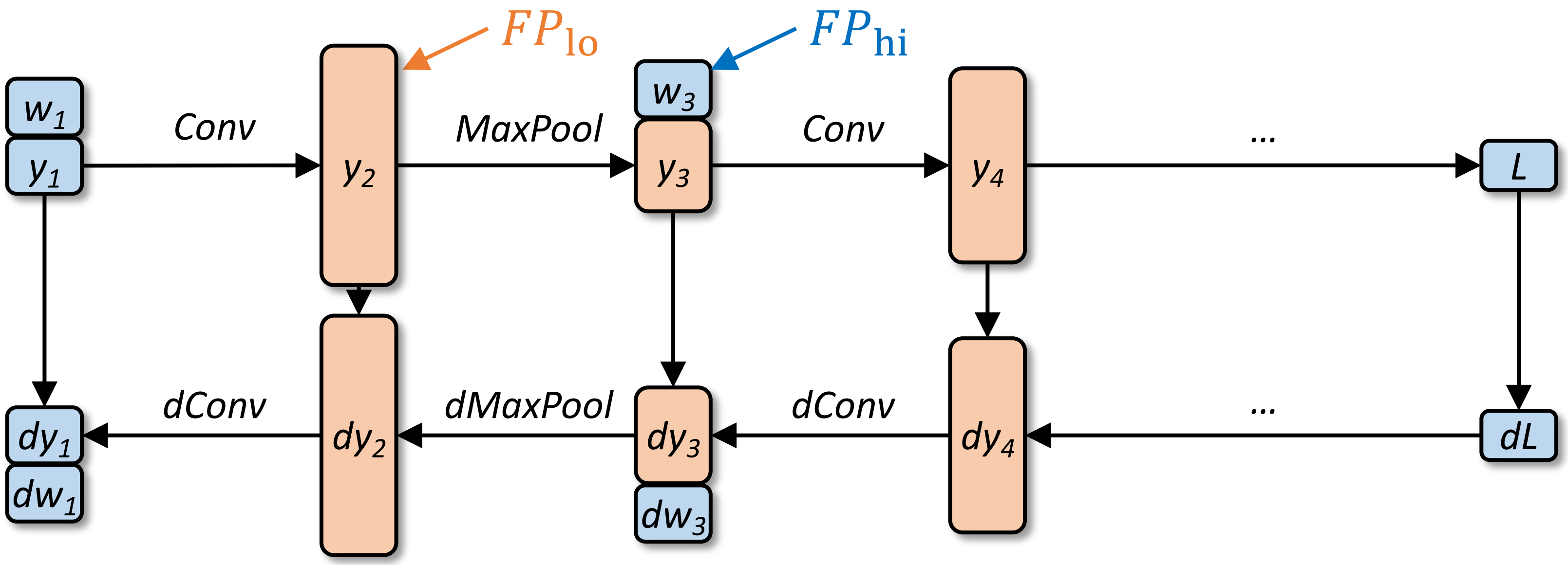

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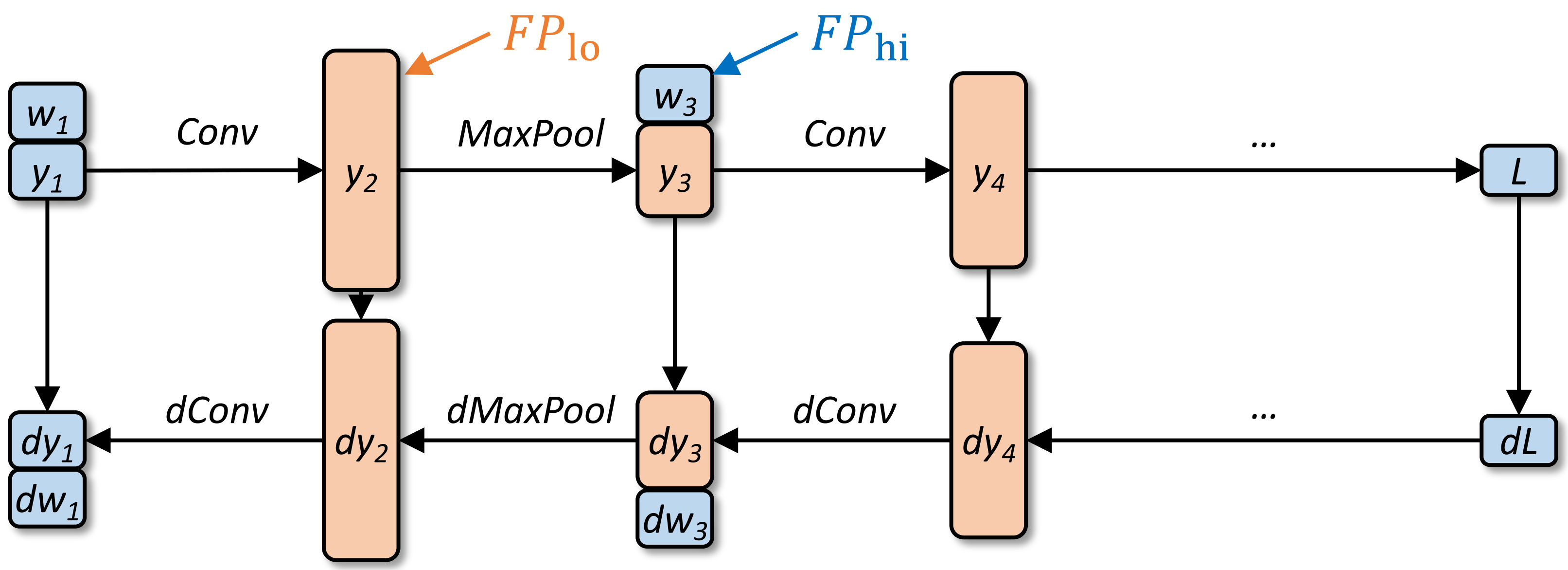
- Optimal in a very simplified setting.
- Empirically better than other orders.

$\pi =$



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- The above method places no explicit constraint on $\boxed{\mathrm{acc}_M(\pi)}$.
  - Observe: Training with this π sometimes diverges, due to too many overflows.

Our Method

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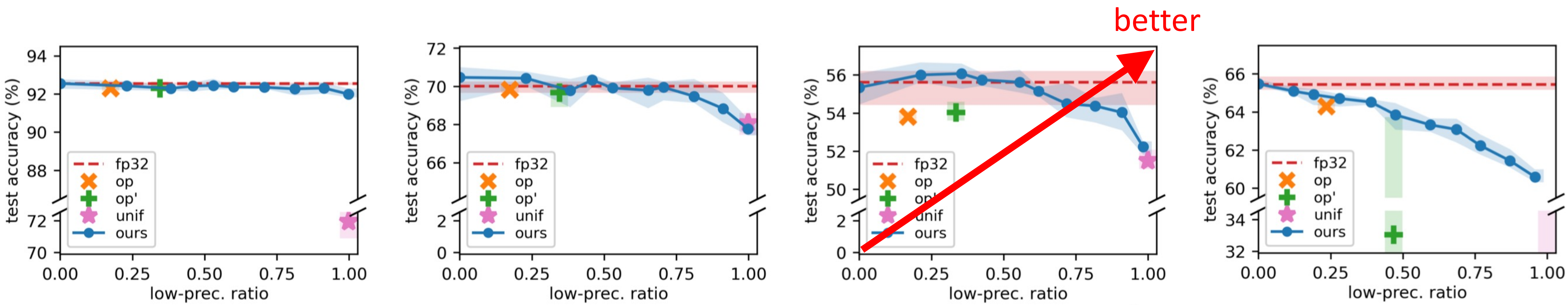
- Our method for **the tradeoff problem:**
    - Initialize π to the all-FP_{hi} assignment.
    - Demote the precision of largest tensors (in size) to FP_{lo}. ⟵ before training
    - Repeat it while $\mathrm{ratio}_{\mathrm{lo}}(\pi) \geq r$. Return the final π.

- The above method places no explicit constraint on $\mathrm{acc}_M(\pi)$.
    - Observe: Training with this π sometimes diverges, due to too many overflows.

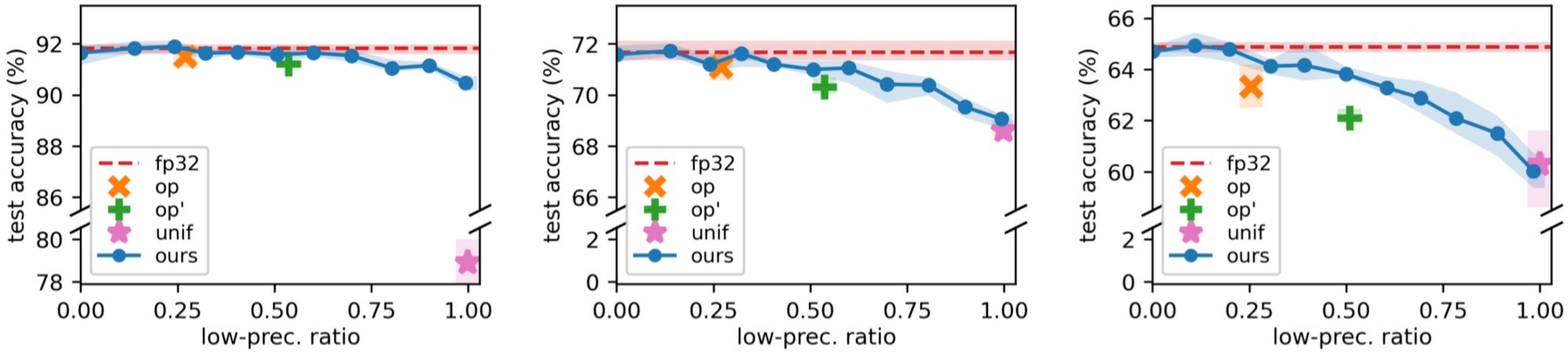
- Our method for **handling overflows:**
    - Promote the precision of tensors that overflow "too much" to FP_{hi}. ⟵ during training

Evaluation Results

- Comparison with existing precision assignments.



(a) CIFAR-10, SqueezeNet (b) CIFAR-100, SqueezeNet (c) CIFAR-100, SqueezeNet† ImageNet, ShuffleNet-v2



(d) CIFAR-10, ShuffleNet-v2 (e) CIFAR-100, ShuffleNet-v2 (f) CIFAR-100, ShuffleNet-v2†

†: width multiplier of 0.25

Summary of Contributions [Submitted]

- We formally introduce the memory-accuracy tradeoff problem and prove its NP-hardness.

- We propose:
  (i) a novel precision assignment method as a heuristic solution to the tradeoff problem;
  (ii) a novel technique that can handle too many overflows arising in training.

- We demonstrate that our techniques outperform existing precision assignments.

Summary of Contributions [Submitted]

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  (i) a novel precision assignment method as a heuristic solution to the tradeoff problem;
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- We demonstrate that our techniques outperform existing precision assignments.

$$\mathrm{NN} \longrightarrow \boxed{\min_{w} \mathcal{L}(\mathrm{NN}_w)}$$

memory & time ⇓

Agenda

Programs that implement `math.h`.	Correctness: [PLDI'16], [POPL'18].
Probabilistic / differentiable programming.	Correctness: [NeurIPS'18/20], [POPL'20/23].
Programs that train ML models.	Acceleration: [Submitted].
Programs that compute derivatives.	Correctness: [ICML'23].

Autodiff

- Autodiff (AD): a class of algorithms that compute

$$\mathcal{D}P(x) \in \mathbb{R}^{m \times n} \quad \text{(when it exists)}$$

  for a given program $P : \mathbb{R}^n \to \mathbb{R}^m$ and input $x \in \mathbb{R}^n$, by applying the chain rule.

- Backpropagation algorithm: an instance of AD, widely used in machine learning.

Correctness of AD

matmul, sequential composition, …

- If P consists of differentiable functions and language constructs, then

$$\exists\ \mathcal{D}P(x) \quad \wedge \quad \mathcal{D}P(x) = \mathcal{D}^{\mathrm{AD}}P(x) \qquad \text{for all } x \in \mathbb{R}^n.$$

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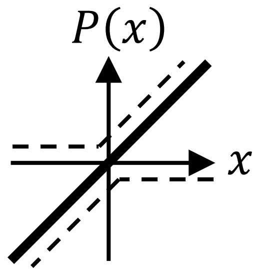
ReLU, if-else statement, …

- If P uses non-differentiable functions or language constructs, then

$$\nexists\, \mathcal{D}P(x) \quad \vee \quad \mathcal{D}P(x) \neq \mathcal{D}^{\mathrm{AD}}P(x) \qquad \text{for some } x \in \mathbb{R}^n.$$

E.g., for $P(x) = \mathrm{ReLU}(x) - \mathrm{ReLU}(-x)$,

$\mathcal{D}P(0) = 1$ but $\mathcal{D}^{\mathrm{AD}}P(0) = 0$.



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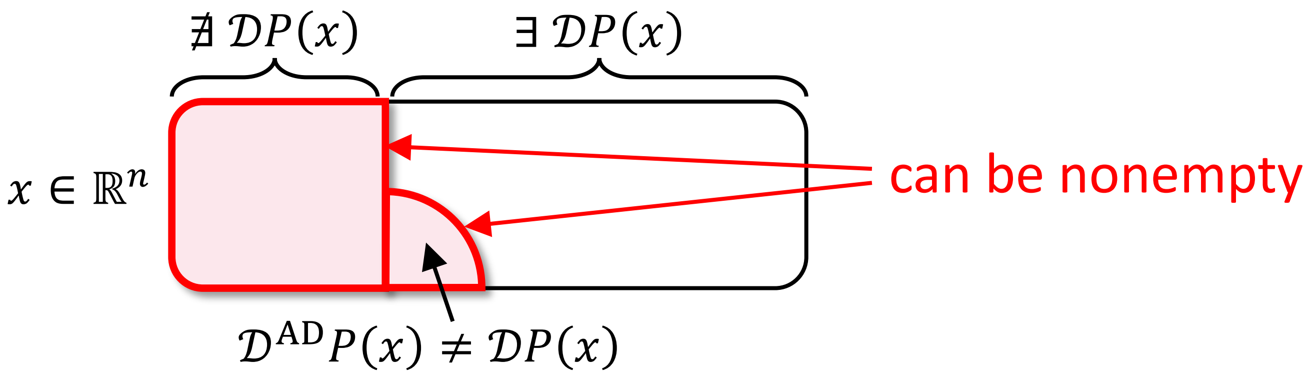
That is,



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My previous result [NeurIPS'20]

"piecewise analytic"

include ReLU, if-else statement, …

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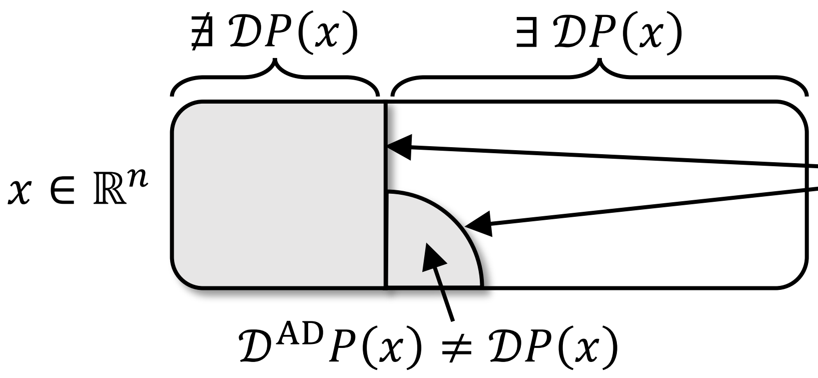


has measure zero (i.e., negligible)

~~can be nonempty~~

Limitations

- If P

My p

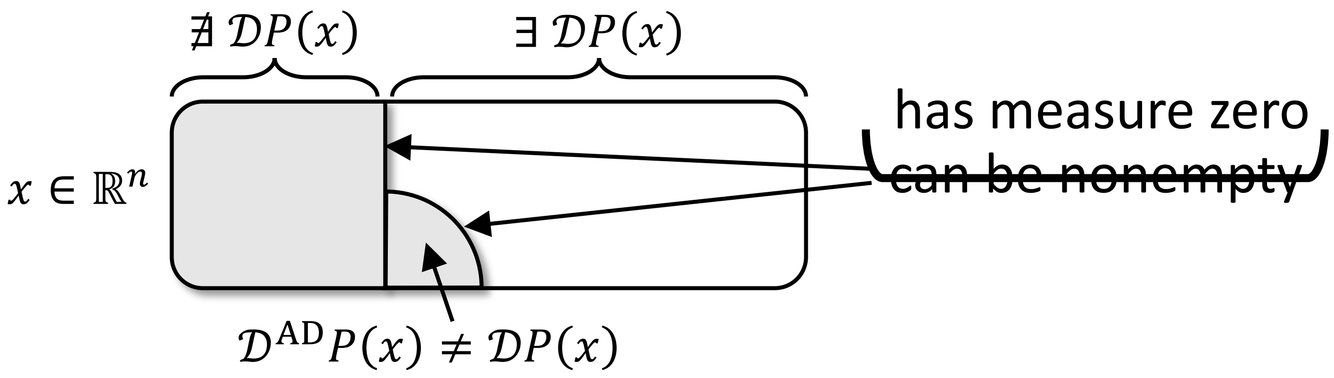
(1) In practice, inputs to programs are not reals, but often floats.

(2) The set of all floats $\mathbb{F}$ is finite, so has measure zero in $\mathbb{R}$.

- If P uses ~~non-differentiable~~ "piecewise analytic" functions or language constructs, then (e.g., ReLU, if-else statement)

$$\nexists\, \mathcal{D}P(x) \quad \vee \quad \mathcal{D}P(x) \neq \mathcal{D}^{\mathrm{AD}}P(x) \qquad \text{for } \cancel{\text{some}}\ \text{“negligible”}\ x \in \mathbb{R}^n.$$

That is,



Limitations

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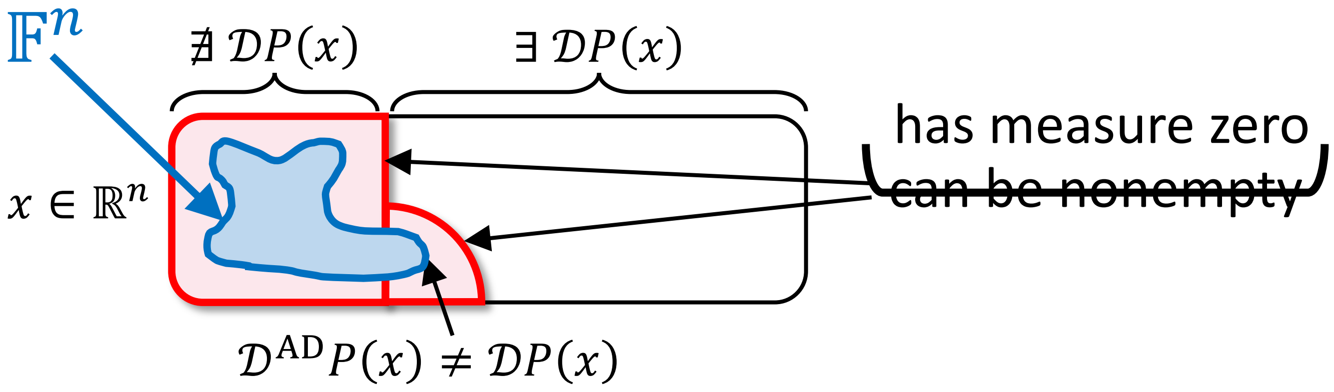
(2) The set of all floats $\mathbb{F}$ is finite, so has measure zero in $\mathbb{R}$.

Hence, AD can be incorrect for all $x \in \mathbb{F}^n$, and this is indeed possible.

That is,



Problem

- Study the correctness of AD when inputs are floats (not reals).
- We focus on programs $P : \mathbb{R}^n \to \mathbb{R}^m$ that represent neural networks:

$$w \quad \longmapsto \quad P(w).$$

Problem

- Study the correctness of AD when inputs are floats (not reals).
- We focus on programs $P : \mathbb{R}^n \to \mathbb{R}^m$ that represent neural networks:

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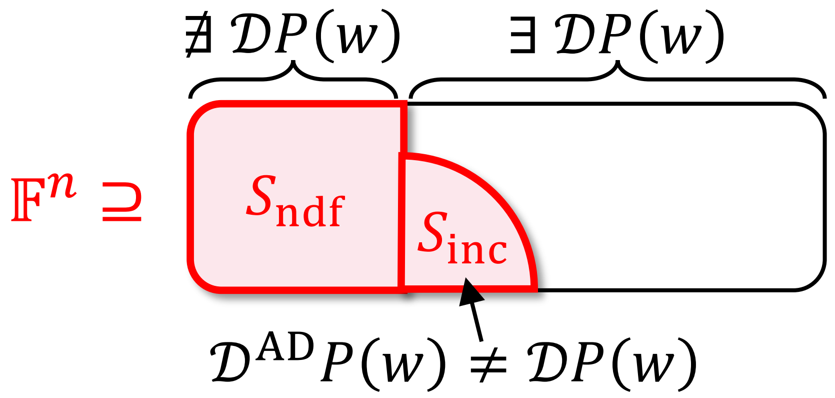
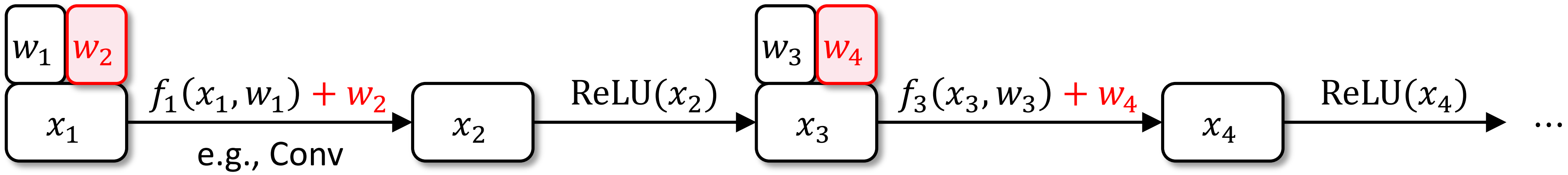
- Goal: Bound the size of the incorrect set (S_{inc}) and non-differentiable set (S_{ndf}) of P.



Our Results

- Consider a neural network P with bias parameters:



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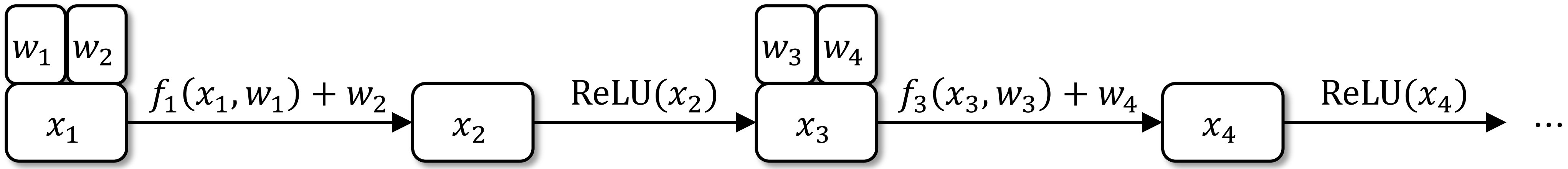
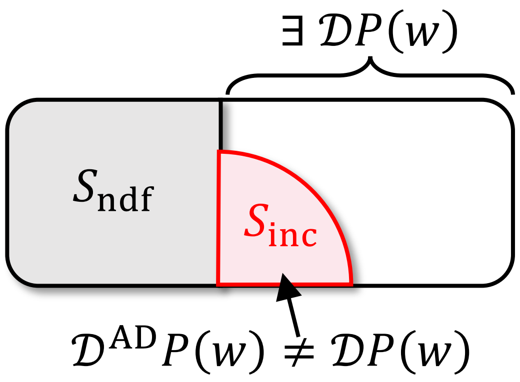


- <u>Theorem 1</u> The incorrect set is always empty:

$$|S_{\mathrm{inc}}| = 0.$$

somewhat surprising, given that
there were no such type of results before



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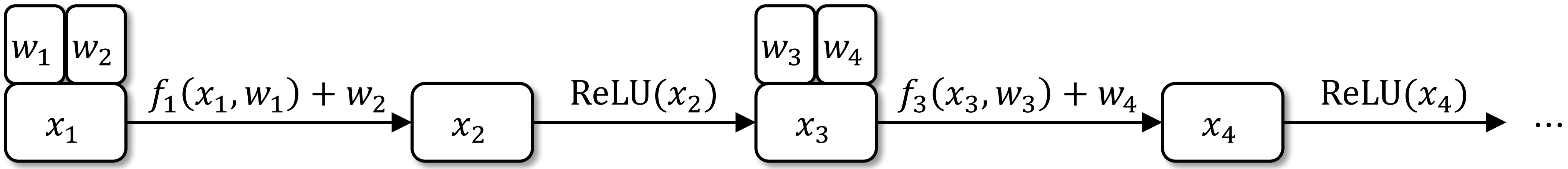
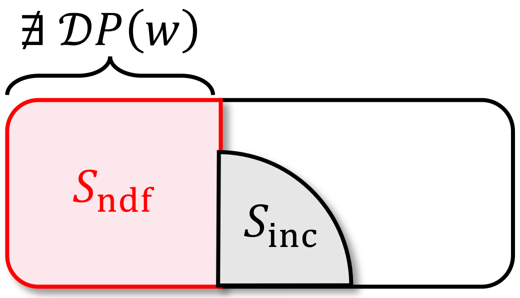


- <u>Theorem 2</u> The density of the non-differentiable set is upper-bounded by

$$\frac{|S_{\mathrm{ndf}}|}{|\mathbb{F}^n|} \leq \frac{\#\ \mathrm{ReLUs\ in}\ P}{|\mathbb{F}|}.$$

2^{32} for 32-bit floats



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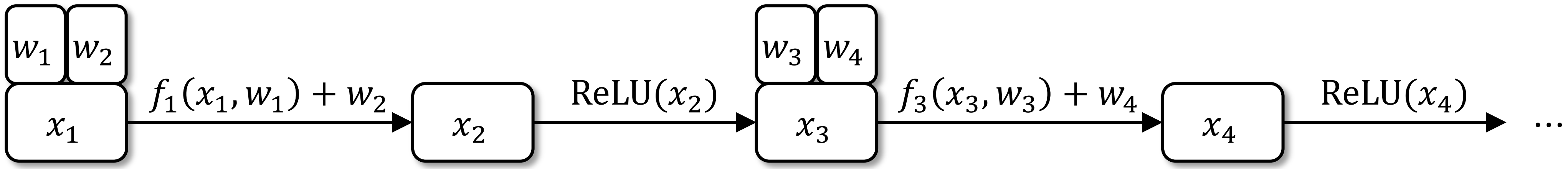

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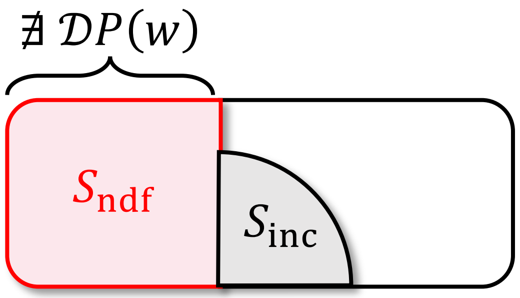

- Theorem 3 For many P, the above density is lower-bounded by

$$\frac{|S_{\mathrm{ndf}}|}{|\mathbb{F}^n|} \geq \frac{1}{2} \cdot \frac{\#\ \mathrm{ReLUs\ in}\ P}{|\mathbb{F}|}.$$

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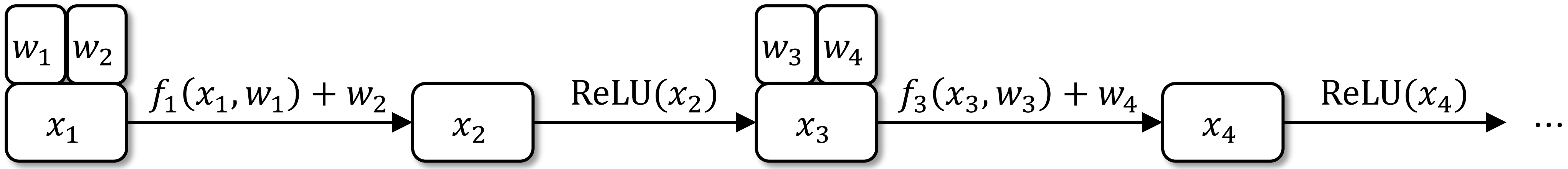
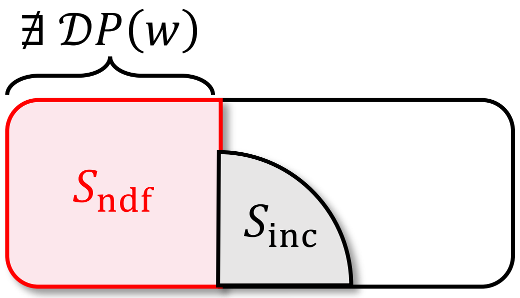


- Theorem 4 Over the non-differentiable set, AD computes a generalized derivative:

$$\mathcal{D}^{\mathrm{AD}} P(w) \in \partial P(w) \qquad \text{for all } w \in S_{\mathrm{ndf}}.$$

Clarke subdifferential of P

$$\triangleq \operatorname{conv}\left\{ \lim_{t\to\infty} \mathcal{D}P(w_t) : w_t \to w \right\}$$



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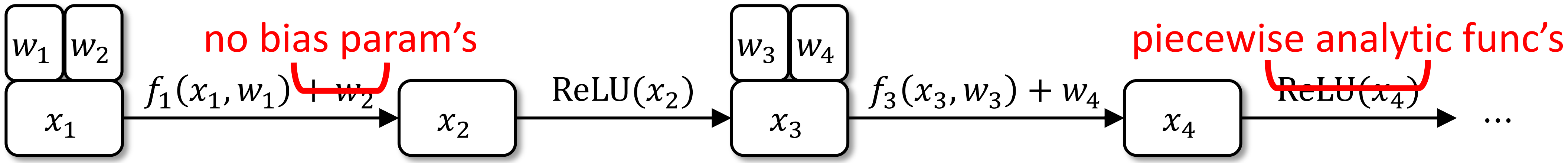

- Theorem 3 On the non-differentiable set, AD computes a generalized derivative:

Extend previous results to more general neural networks.

Main point: Bounds become larger without bias parameters.

Summary of Contributions [ICML'23]

- We theoretically study the correctness of AD for neural networks when param's are floats.
- We prove tight bounds on the density of the incorrect and non-differentiable sets.
  We also prove what AD computes over these sets.
- Our results imply that AD for neural networks is correct on most floating-point param's,
  and it is correct more often with bias parameters.

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$$P \longrightarrow \boxed{\cdots \ \mathcal{D}^{\mathrm{AD}} P \ \cdots}$$

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$\Longrightarrow$ Have widened our understanding of floating point in real-world systems.

Questions?