

Floating-Point Neural Networks Are Provably Robust Universal Approximators

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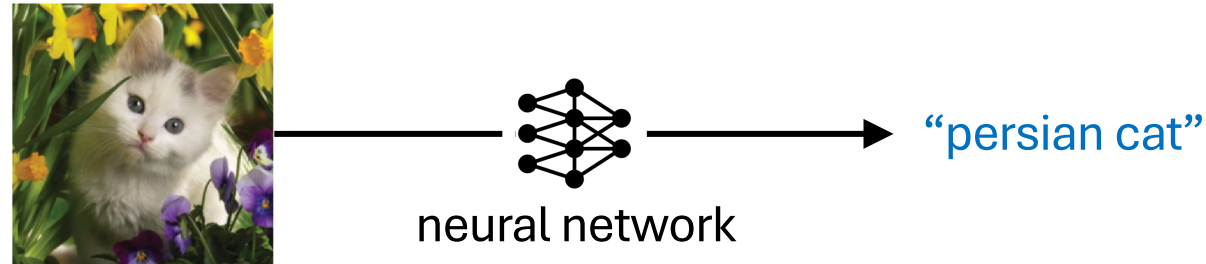
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CAV, July 2025

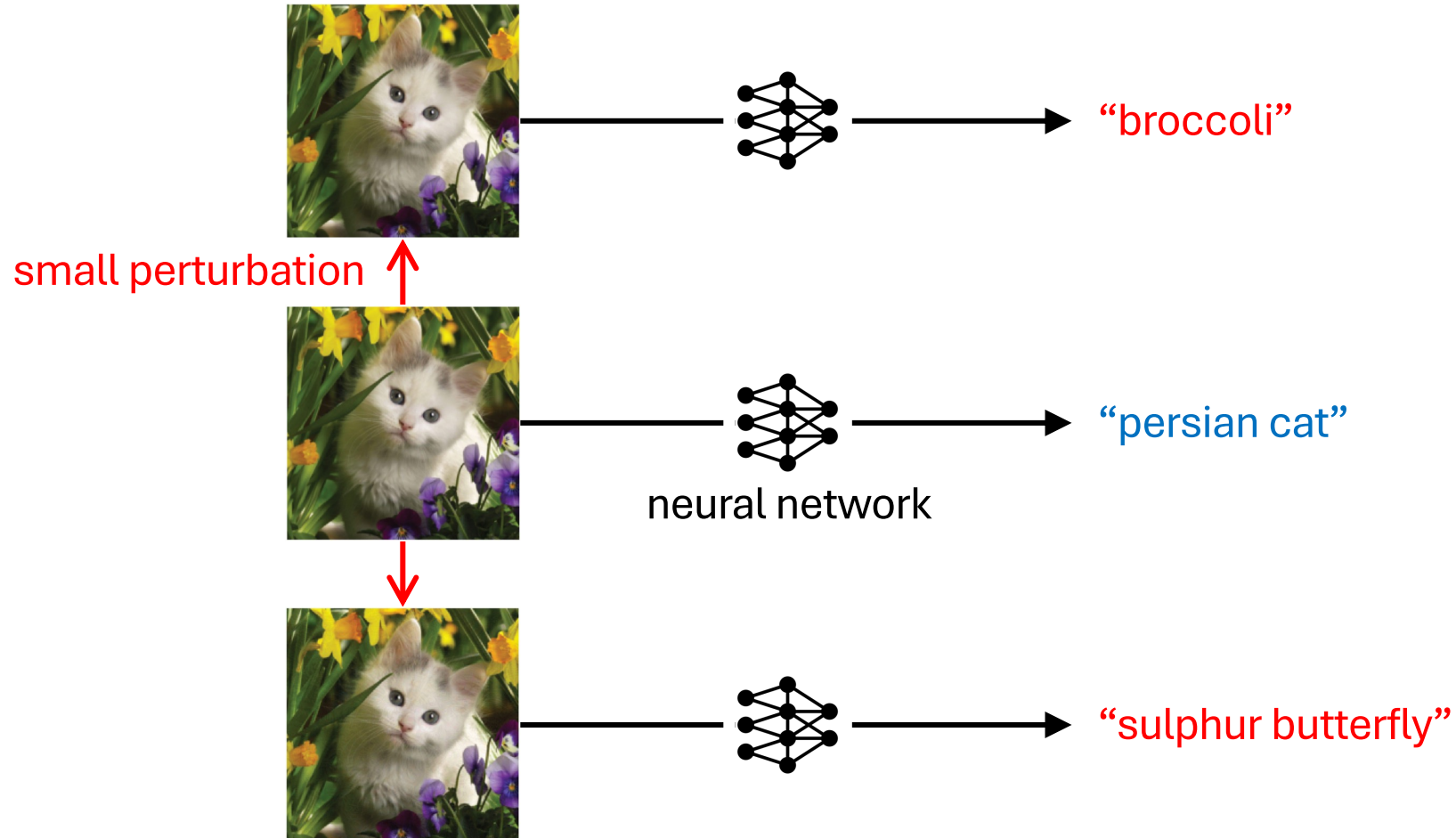
Robustness Issue of Neural Networks

- **Neural networks** can do amazing things.



Robustness Issue of Neural Networks

- Neural networks can do amazing things. But they are often **not robust**.



Provably Robust Neural Networks

- Many techniques have been developed to **ensure the robustness** of NNs.
 - **Robustness verification:** Prove the robustness of a given NN.
 - **Robust training:** Train a new NN such that it is provably robust (and performs well).

Reluplex: An Efficient SMT Solver for Verifying Deep Neural Networks

Guy Katz, Clark Barrett, David Dill, Kyle Julian and Mykel Kochenderfer

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Differentiable Abstract Interpretation for Provably Robust Neural Networks

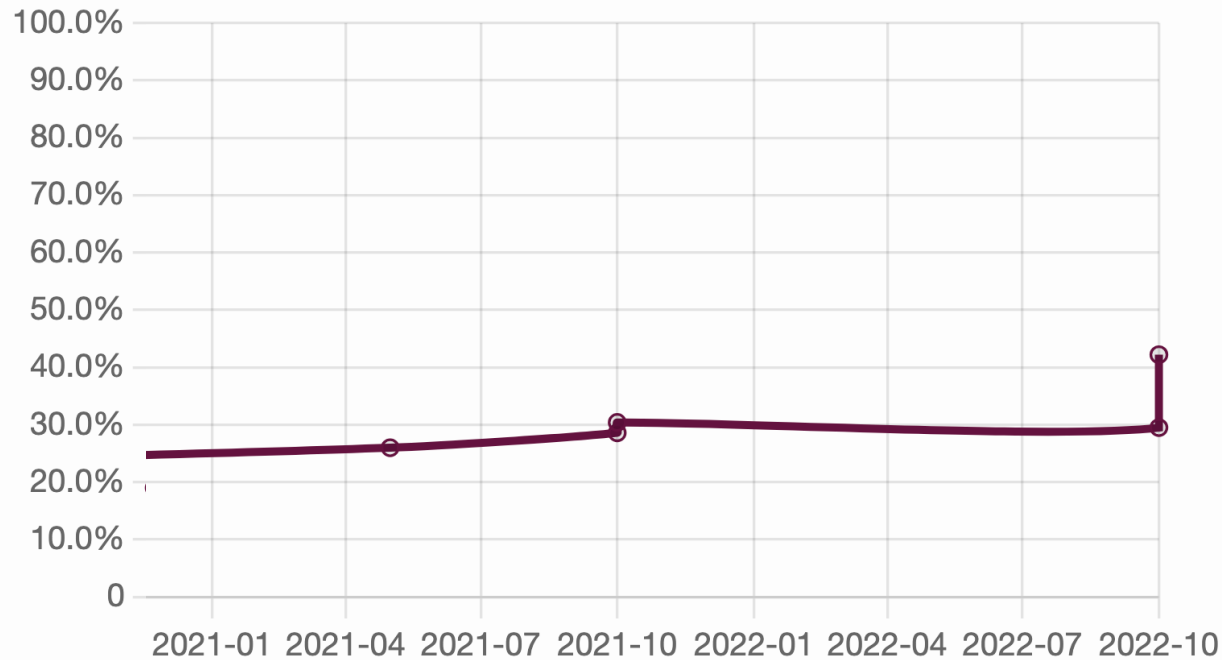
Matthew Mirman¹ Timon Gehr¹ Martin Vechev¹

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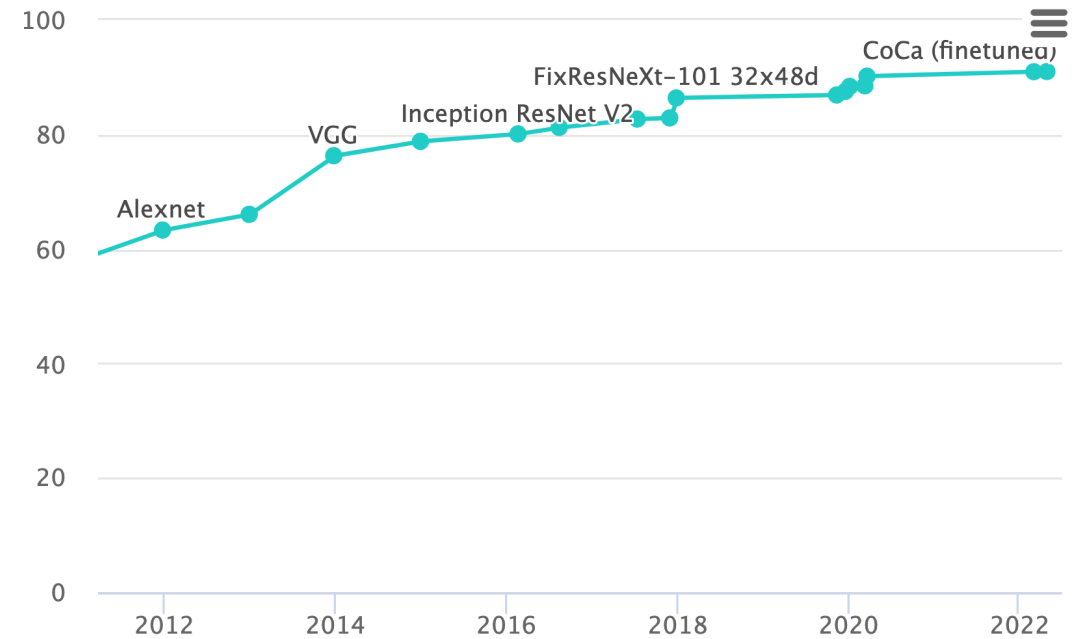
Provably Robust Neural Networks

- **Provably robust NNs still fail to achieve** the **state-of-the-art accuracy**.

Image classification on ImageNet



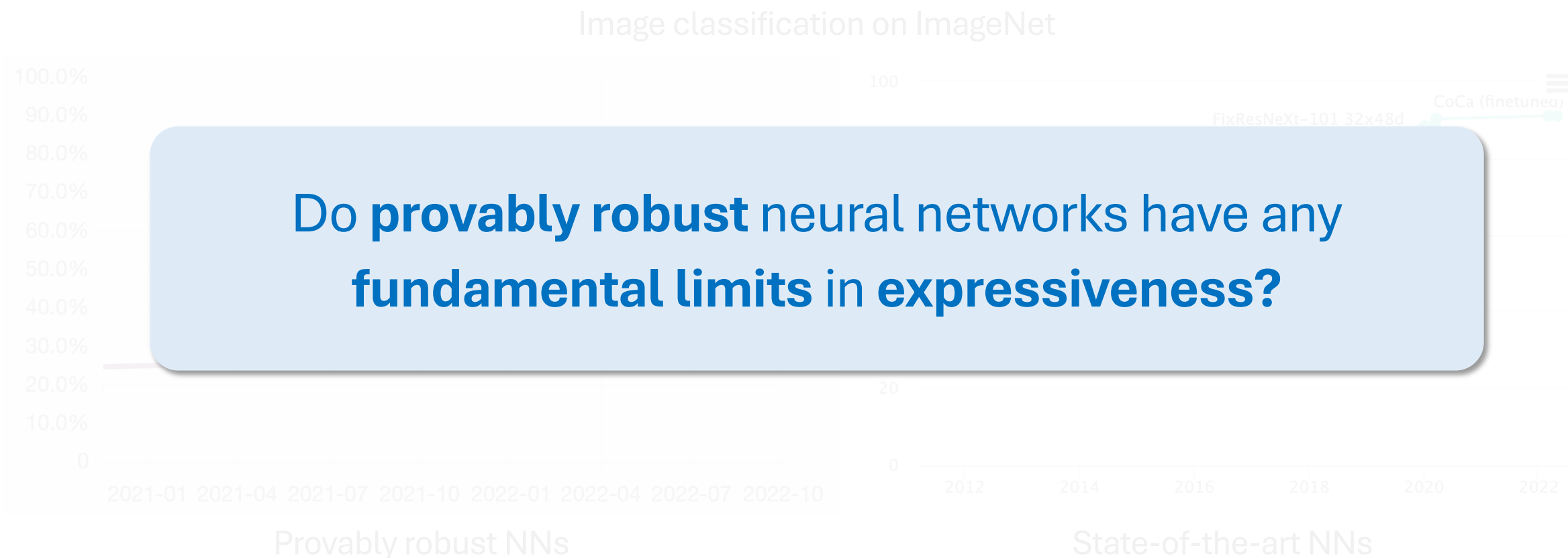
provably robust NNs



state-of-the-art NNs
(not provably robust)

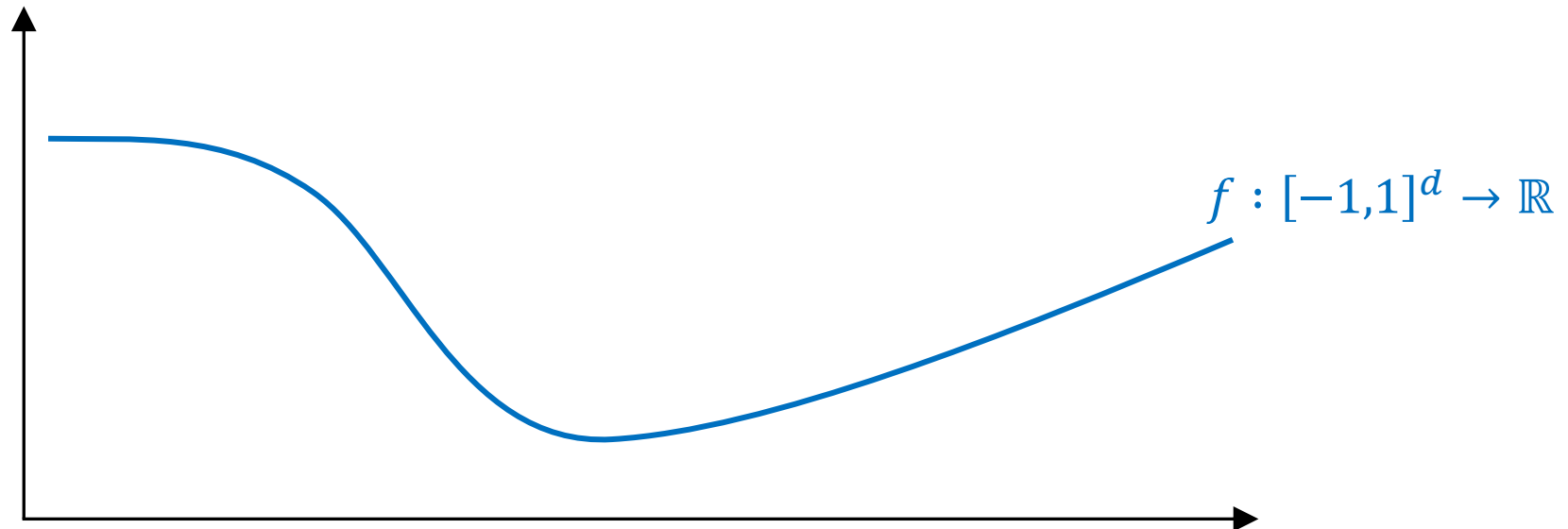
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Expressiveness of ~~Provably Robust~~ Neural Networks

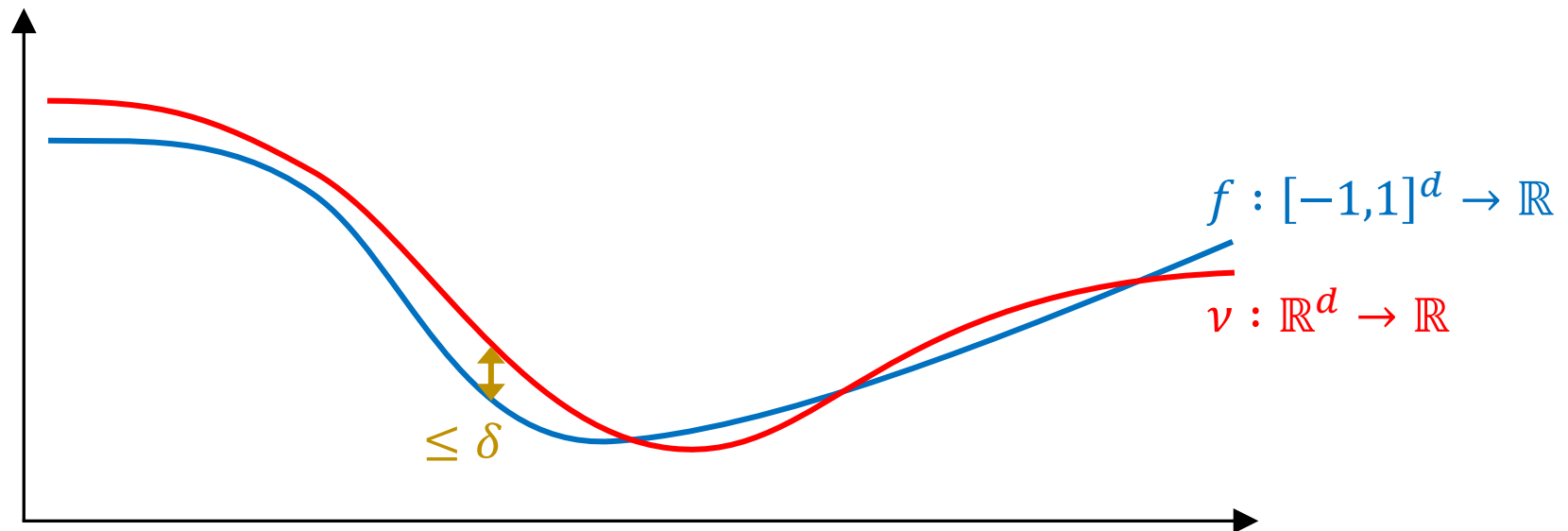
- **No fundamental limit** exists by **universal approximation (UA)** theorems.
- **Theorem.** $f : [-1,1]^d \rightarrow \mathbb{R}$... target func (continuous).
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For any $\delta > 0$, there exists a σ -neural network v : $\mathbb{R}^d \rightarrow \mathbb{R}$ such that

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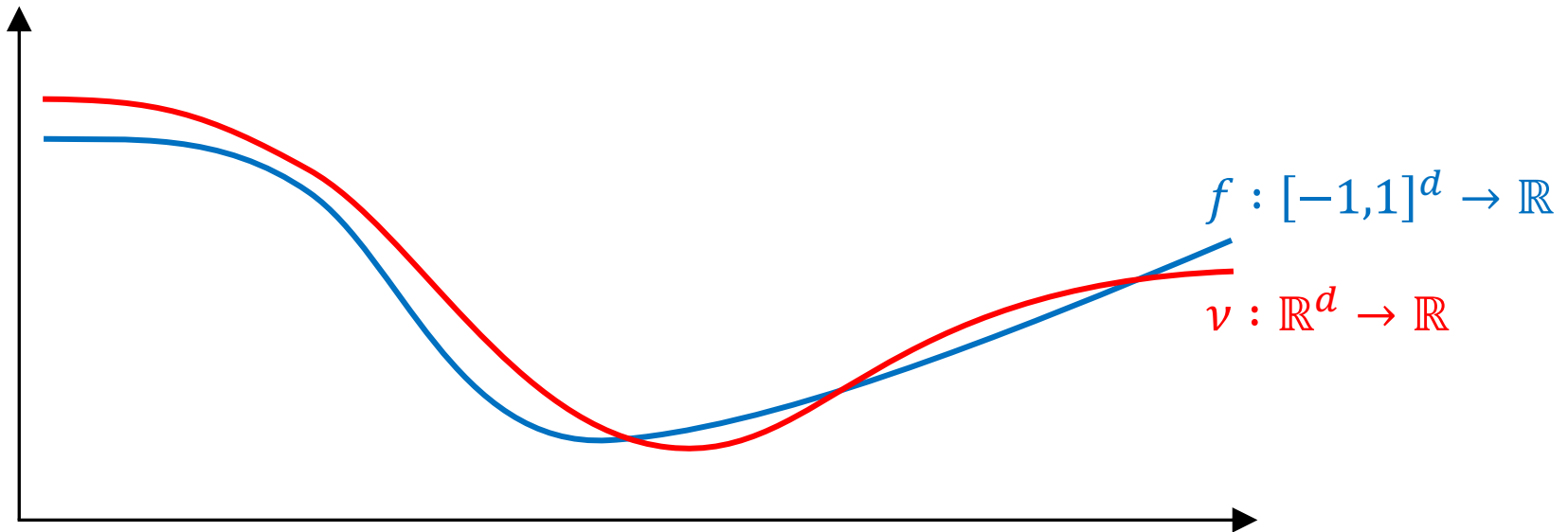
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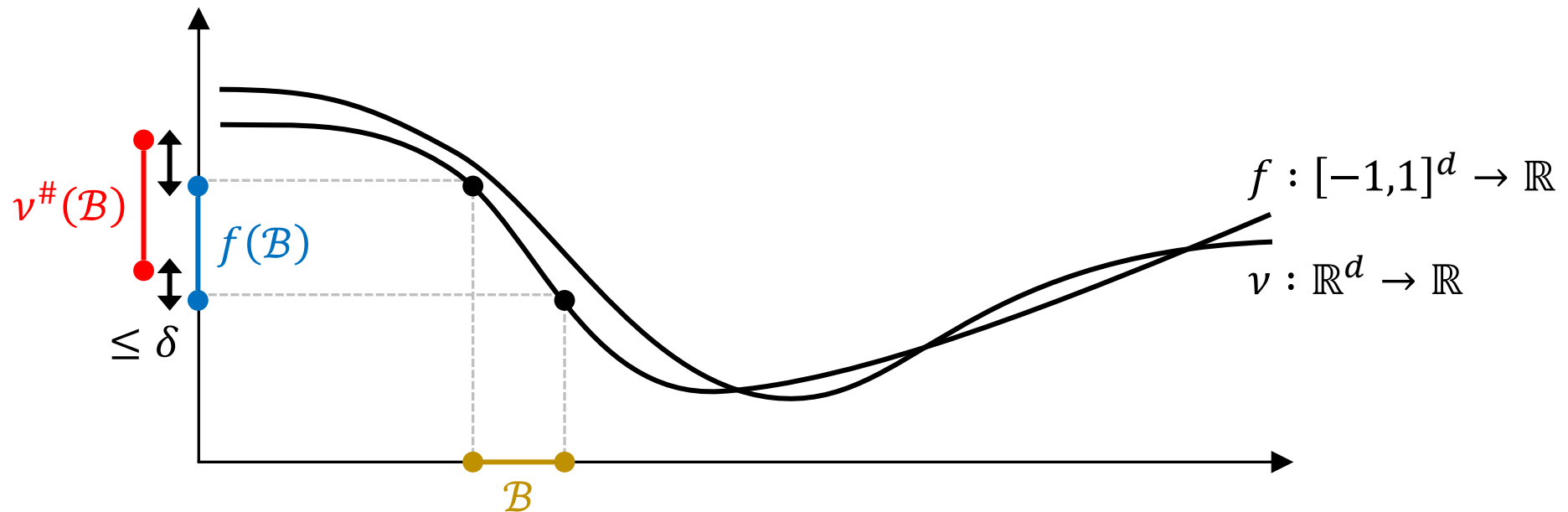
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\exists ideal classifier h (not NN) that is robust (not provably robust)
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“Provably robust NNs have **no fundamental limits in expressiveness.**”

Limitation of Existing Results

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Fundamental limits **may still exist** in practice for provably robust NNs.

Our Work: Overview

Do existing results still hold in **real-world settings**?

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Do existing results still hold in **real-world settings**?

We study the *expressiveness of provably robust NNs* **over floats**.

- Prove the *IUA theorem* **over floats**.
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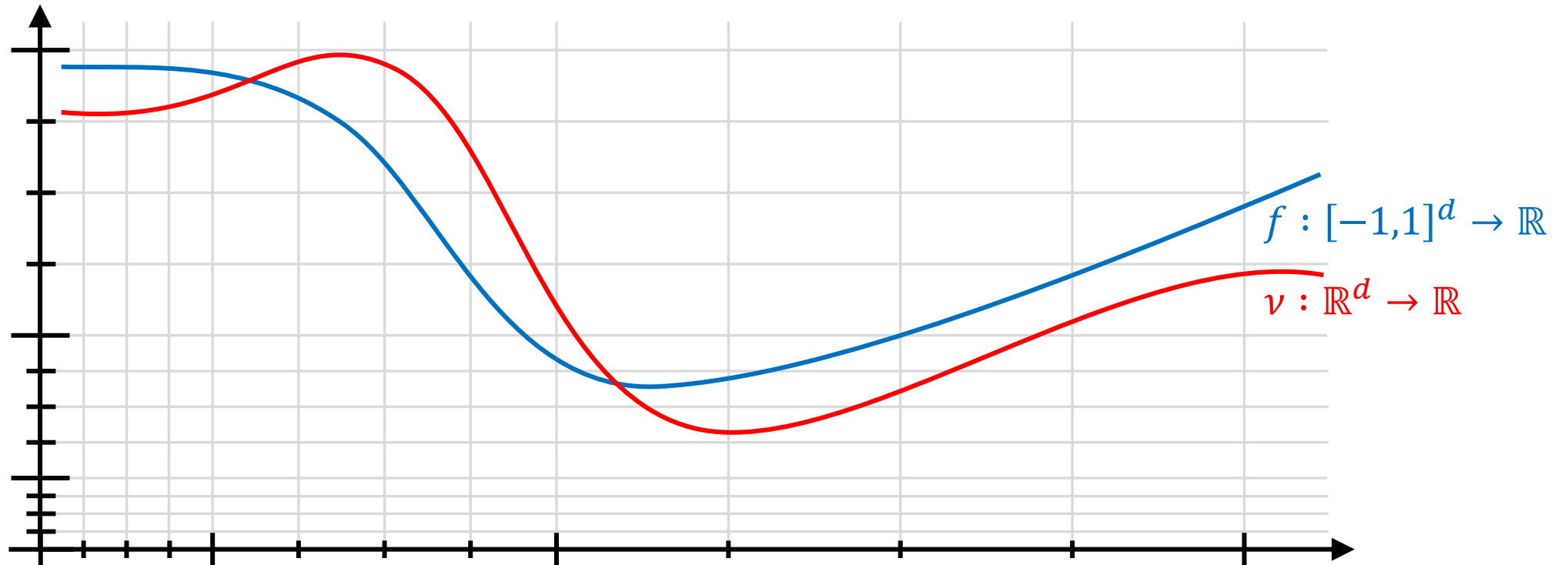
- Prove the *IUA theorem* **over floats**.
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- Prove the *computational completeness of “simple” programs* **over floats**.

Our Main Results

What to Prove?

- **Theorem.** Let $f : [-1,1]^d \rightarrow \mathbb{R}$ and $\delta > 0$. defined using exact arithmetic
There exists a σ -neural network $v : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

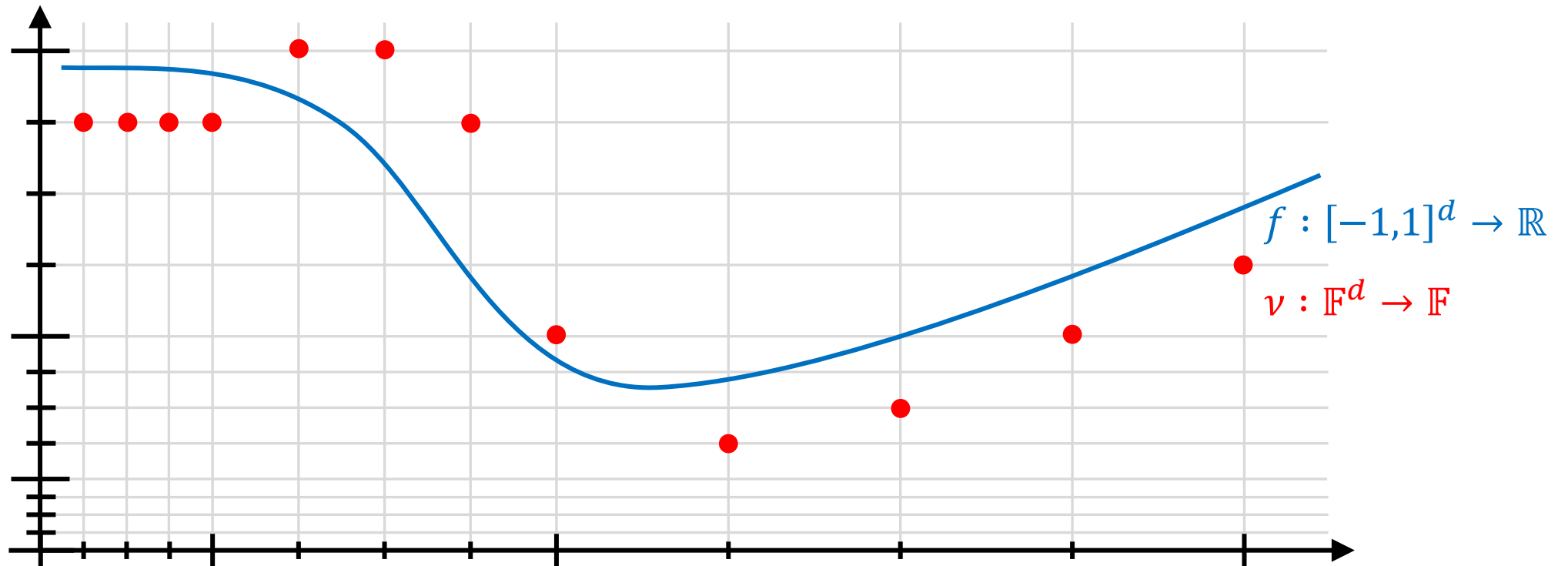
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What to Prove?

- **Theorem?** Let $f : [-1,1]^d \rightarrow \mathbb{R}$ and $\delta > 0$. defined using FP arithmetic
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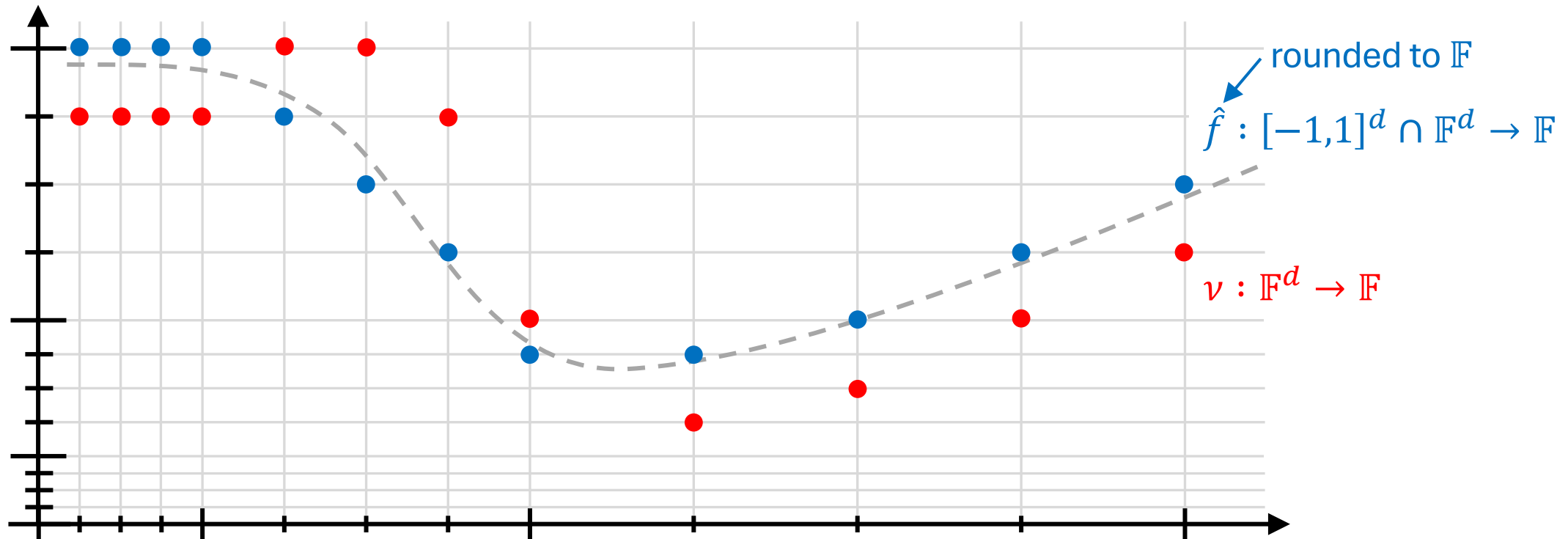


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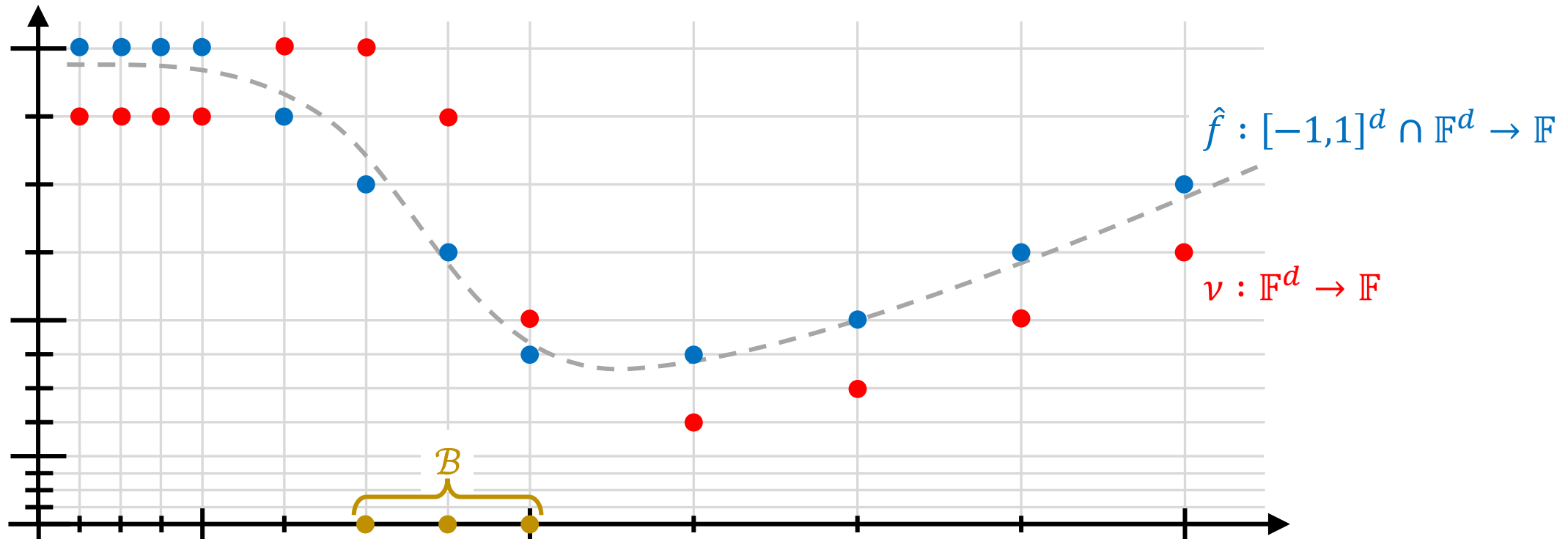


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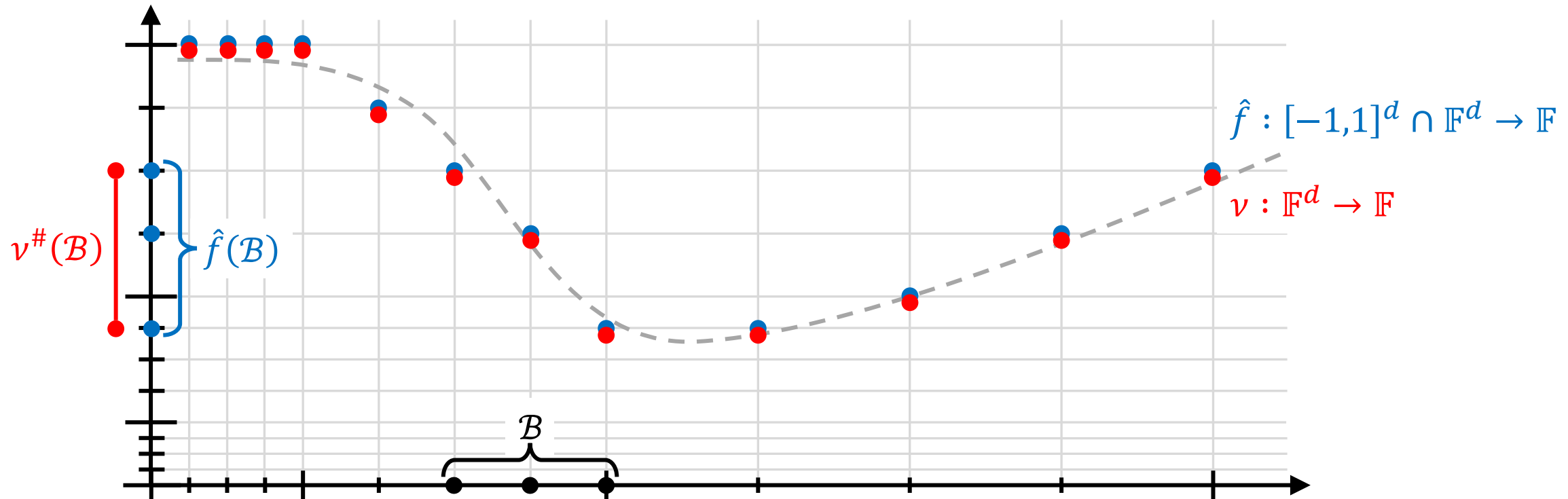
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• **Theorem!** Let $f : [-1,1]^d \rightarrow \mathbb{R}$.

There exists a σ -neural network $\nu : \mathbb{F}^d \rightarrow \mathbb{F}$ such that

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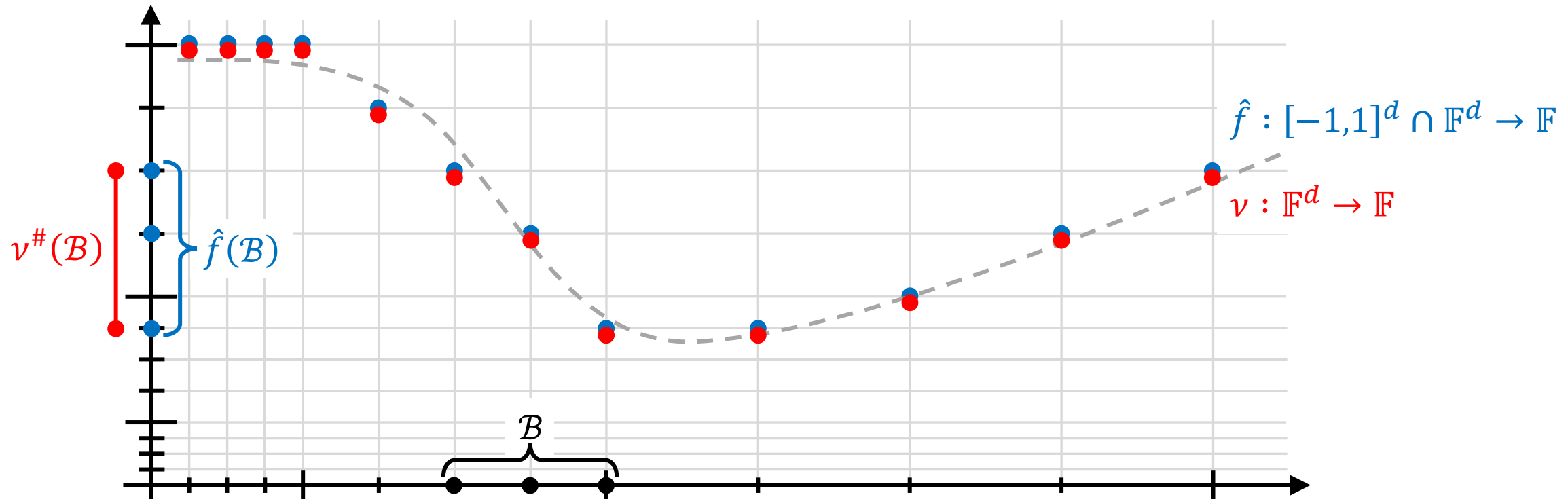


IUA Theorem Over \mathbb{F}

- **Theorem!** Let $f : [-1,1]^d \rightarrow \mathbb{R}$. Assume $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ satisfies **mild conditions**.

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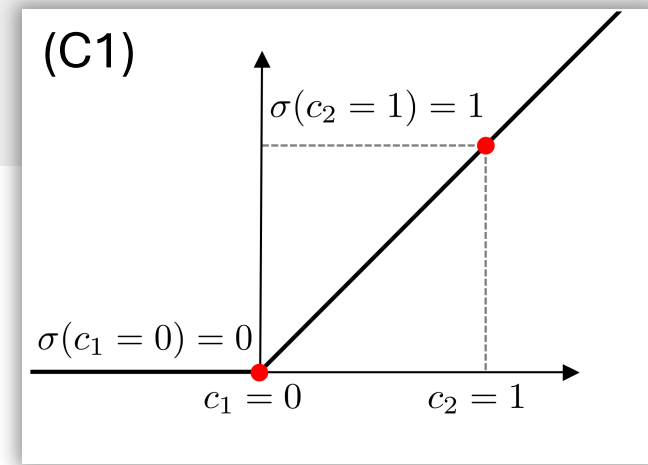
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- **Conditions on $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ (Informal).**

(C1) $\exists c_1, c_2 \in \mathbb{F}$ such that $\sigma(c_1) = 0$ and $\frac{\varepsilon}{2} \leq |\sigma(c_2)| \leq \frac{5}{4}$.



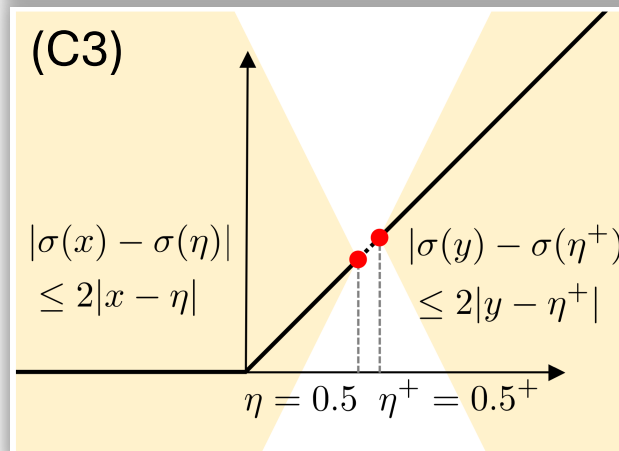
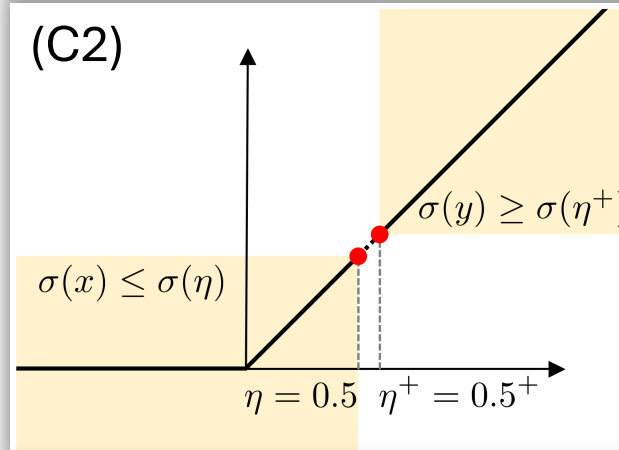
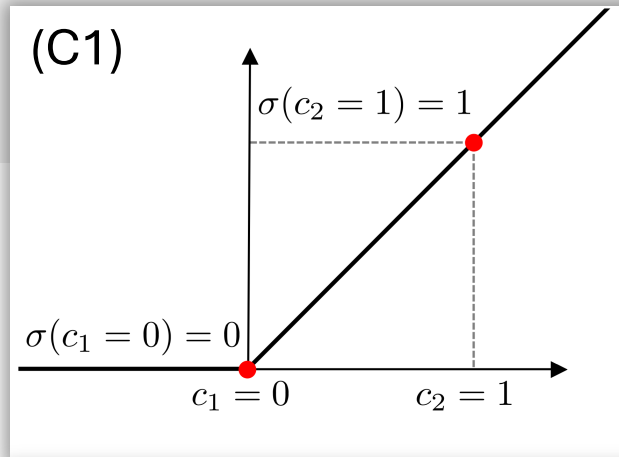
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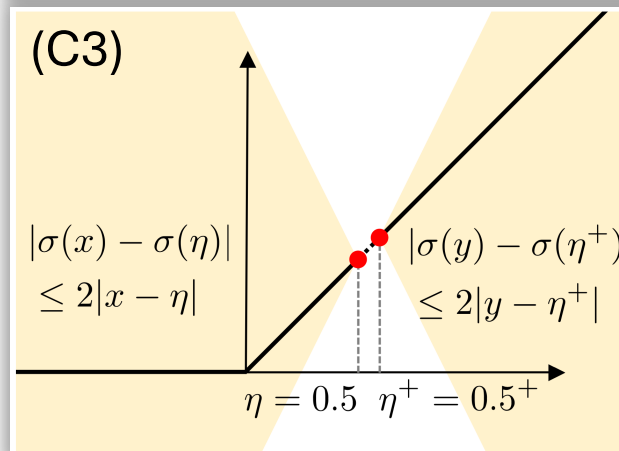
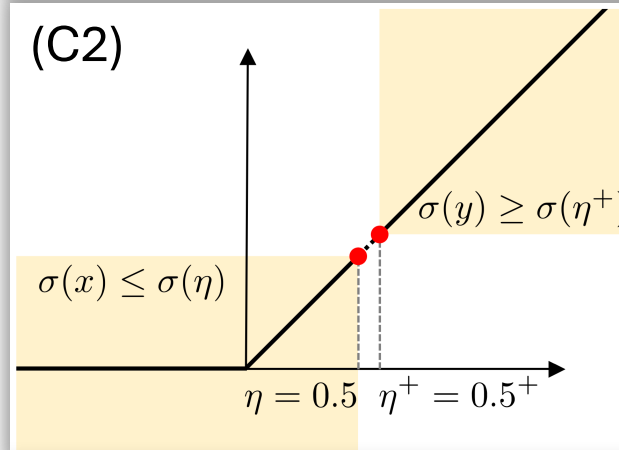
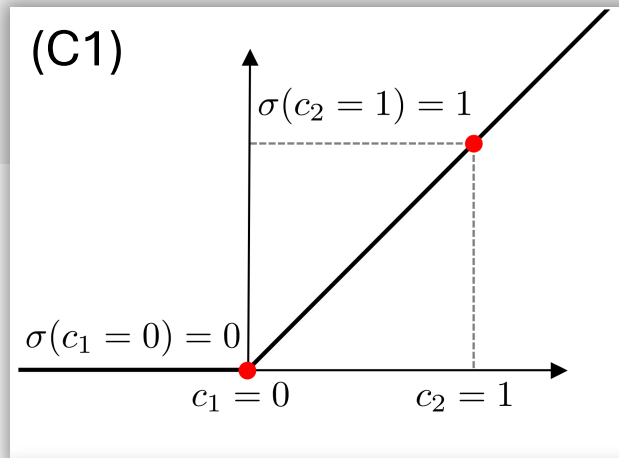
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- **Proposition.** The correct roundings of the following activation func's satisfy the conditions (C1)--(C3):

ReLU, LeakyReLU, ELU, GELU, Mish, softplus, sigmoid, tanh : $\mathbb{R} \rightarrow \mathbb{R}$.



IUA Theorem Over \mathbb{R} vs. \mathbb{F}

- **Approximation Power.**

- Over \mathbb{R} : NNs can **sufficiently** approximate **continuous** target functions ($\mathbb{R} \rightarrow \mathbb{R}$).
- Over \mathbb{F} : NNs can **exactly** compute **any** target functions ($\mathbb{F} \rightarrow \mathbb{F}$).

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 - σ -NN over \mathbb{F} can be **non-affine over \mathbb{R}** ($\because \text{aff}_{\mathbb{F}} : \mathbb{F}^k \rightarrow \mathbb{F}$ are often **non-affine over \mathbb{R}** by rounding error).

Implications of Our IUA Theorem

Provable Robustness Over \mathbb{F} .

- **Theorem (Informal).** \exists ideal classifier f over \mathbb{F} (not NN) that is robust (not provably robust)
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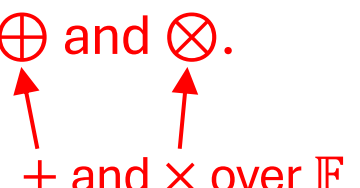
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“**Provably robust NNs** over \mathbb{F} have **no fundamental limits** in expressiveness.”

- **Note.** **Positive answer** to the main question raised earlier in this talk.

Implications of Our IUA Theorem

Computational Completeness Over \mathbb{F} .

- **Theorem (Informal).** All **terminating** programs that take and return **floats** can be expressed by **straight-line** programs using **only** \oplus and \otimes .


The diagram consists of two red arrows pointing upwards. The left arrow points from the text '+ and x over F' to the symbol \oplus . The right arrow points from the same text to the symbol \otimes .

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Prove this theorem by **extending our IUA theorem for $\sigma = \text{id}$** .

Summary

Provably robust NNs have no fundamental limit in expressiveness, even over floats.

- Prove the **IUA theorem for NNs over \mathbb{F}** .
- Prove the **existence of provably robust NNs over \mathbb{F}** .

Summary

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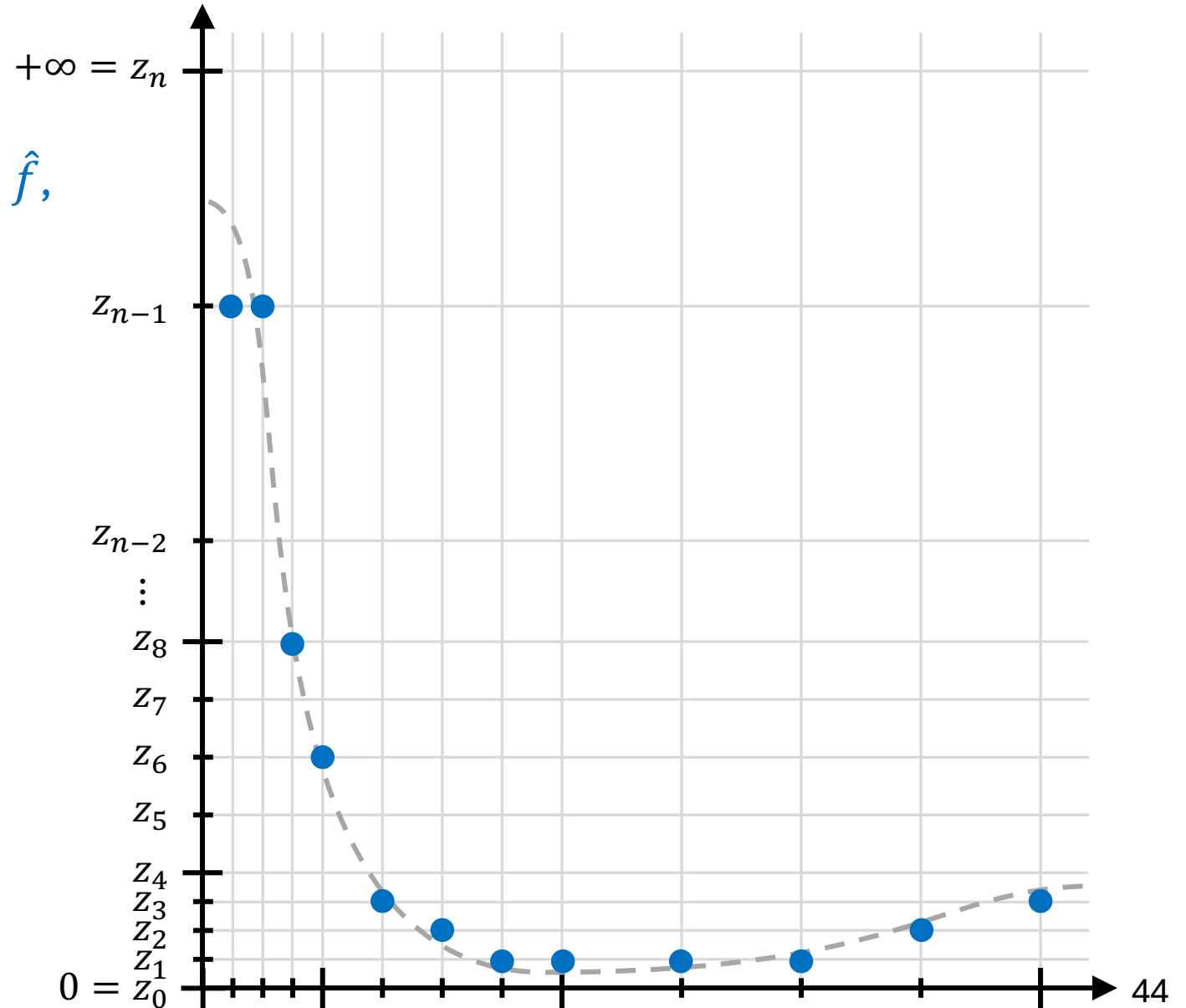
Unexpected byproducts.

- Identify **fundamental distinctions** between two computations models: **over \mathbb{F} and over \mathbb{R}** .
- Prove that **all halting programs over \mathbb{F}** can be expressed using only two operations: \oplus and \otimes .

IUA Theorem Over \mathbb{F}

- **Proof Sketch.**

To approximate the rounded **target function** \hat{f} ,
we “stack” **indicator functions**.

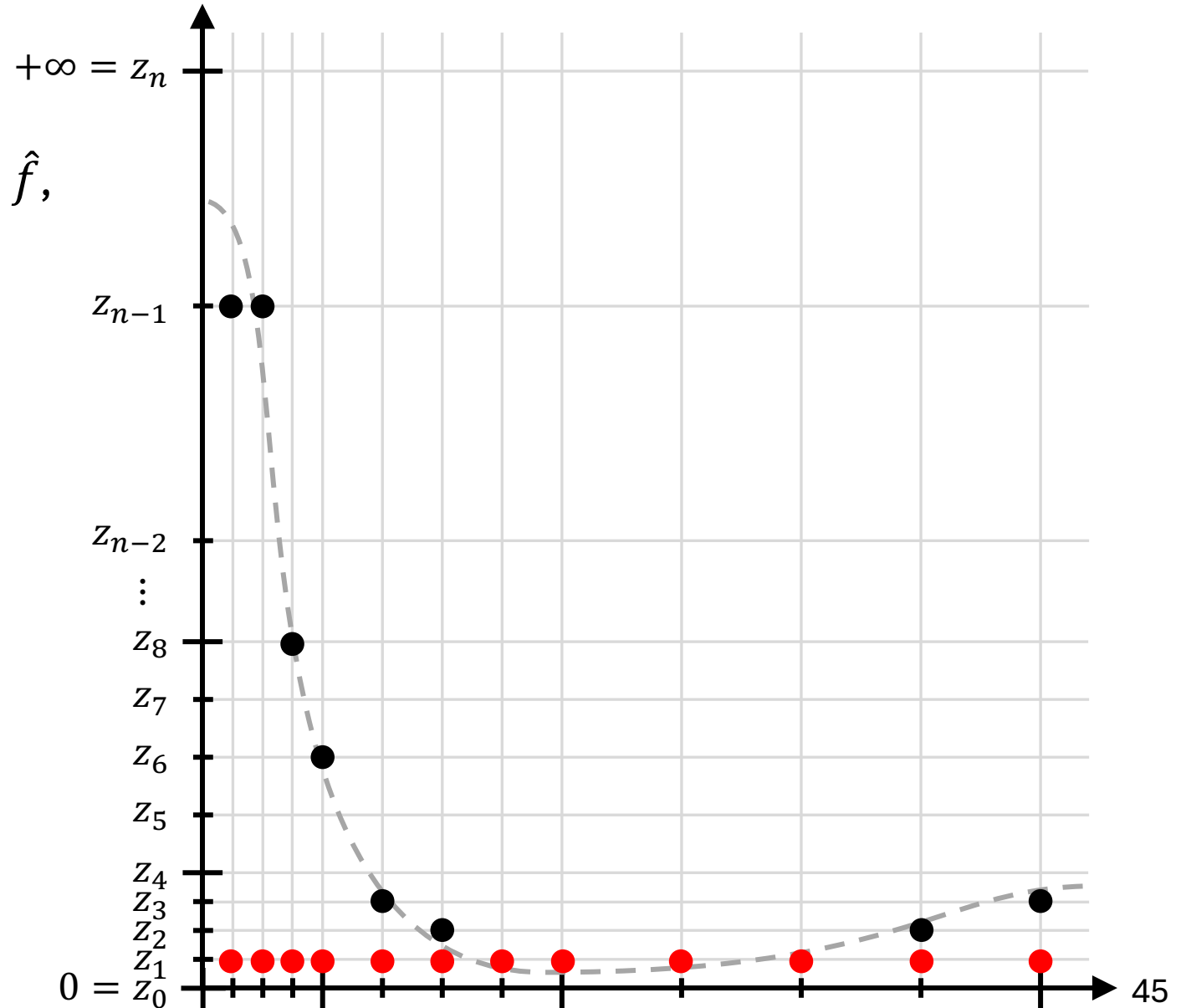


IUA Theorem Over \mathbb{F}

- **Proof Sketch.**

To approximate the rounded target function \hat{f} , we “stack” indicator functions.

$$v(x) = (z_1 \ominus z_0) \otimes \mathbb{1}[\hat{f}(x) \geq z_1]$$

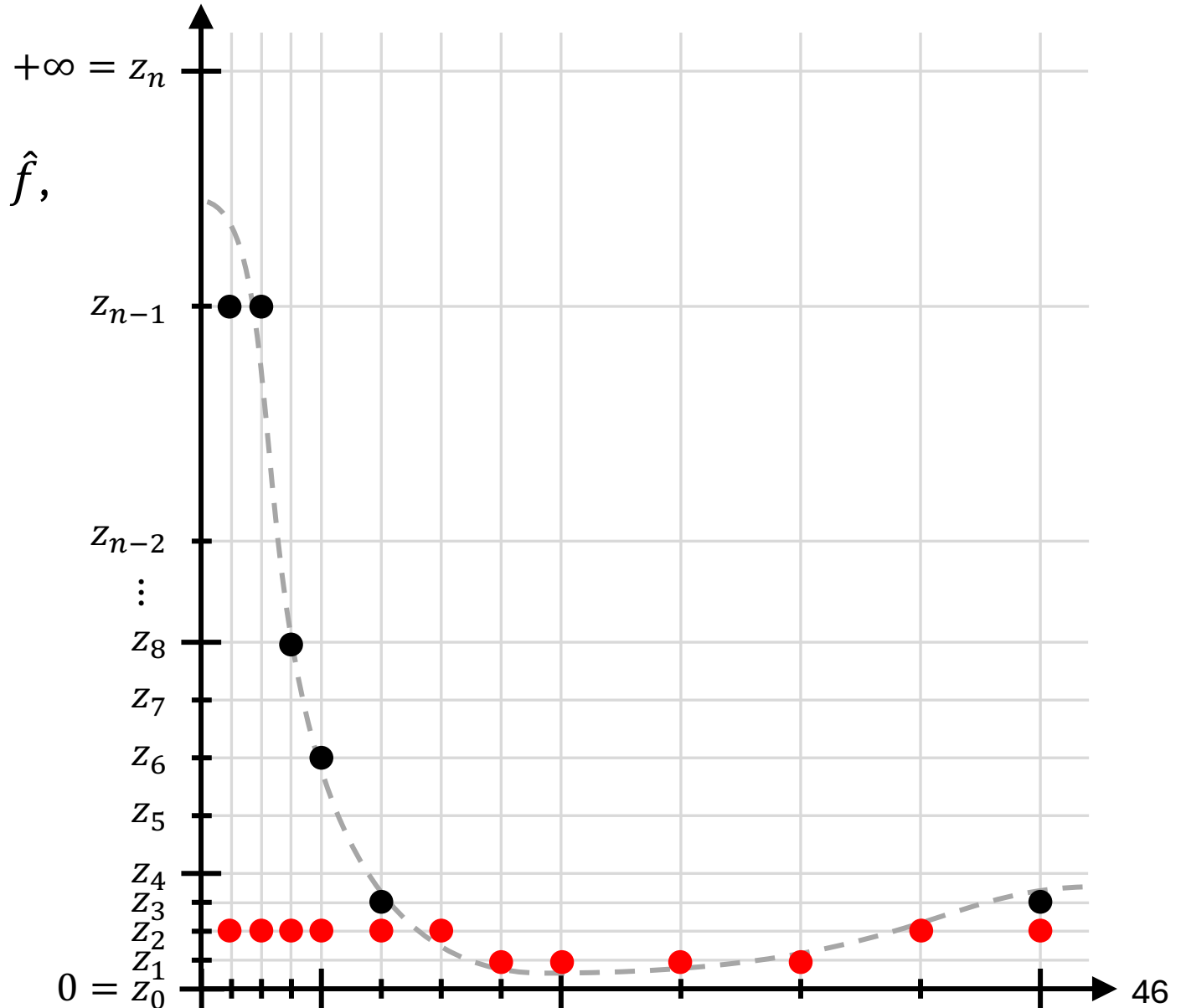


IUA Theorem Over \mathbb{F}

- **Proof Sketch.**

To approximate the rounded target function \hat{f} , we “stack” indicator functions.

$$v(x) = (z_1 \ominus z_0) \otimes \mathbb{1}[\hat{f}(x) \geq z_1] \\ \oplus (z_2 \ominus z_1) \otimes \mathbb{1}[\hat{f}(x) \geq z_2]$$

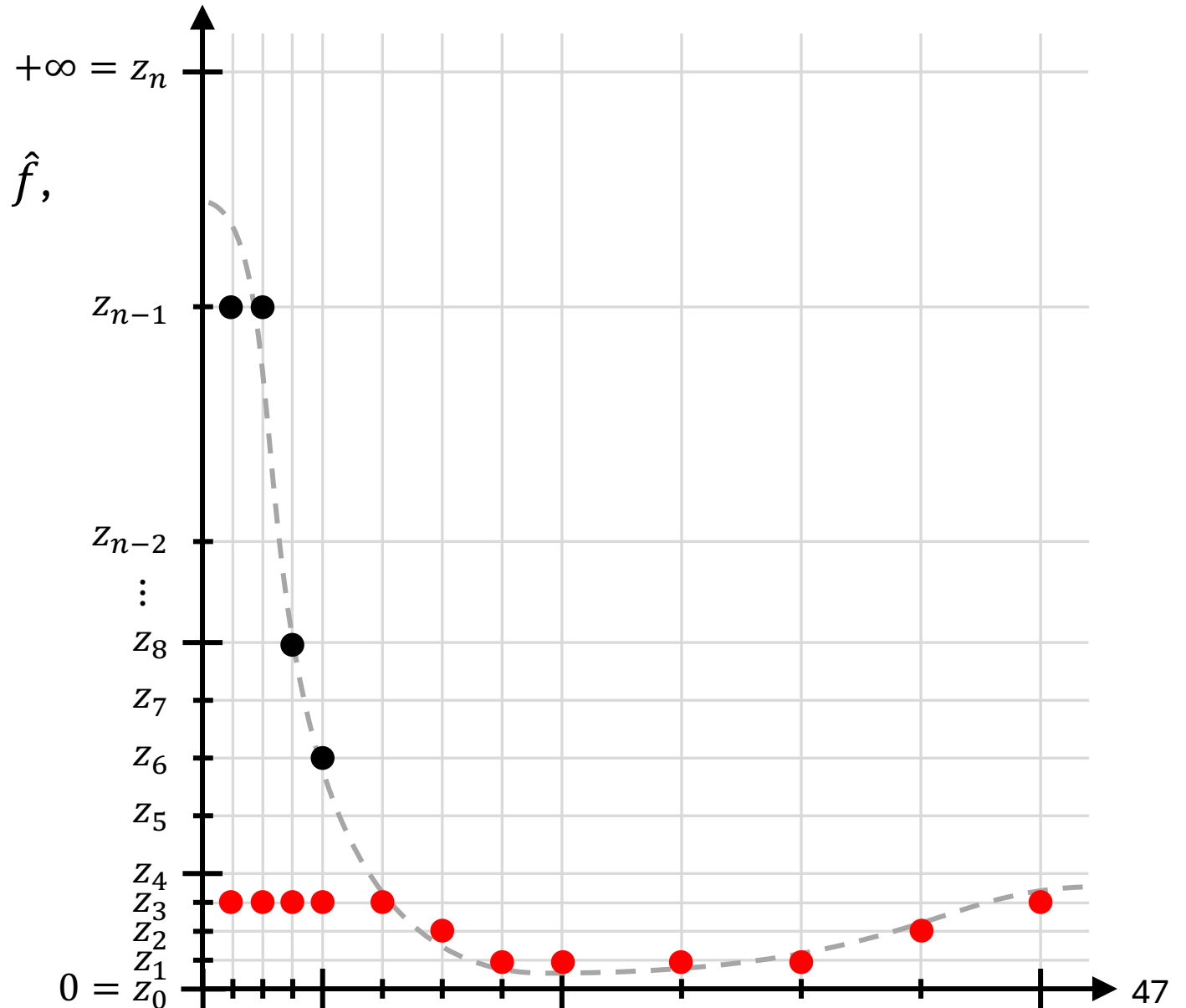


IUA Theorem Over \mathbb{F}

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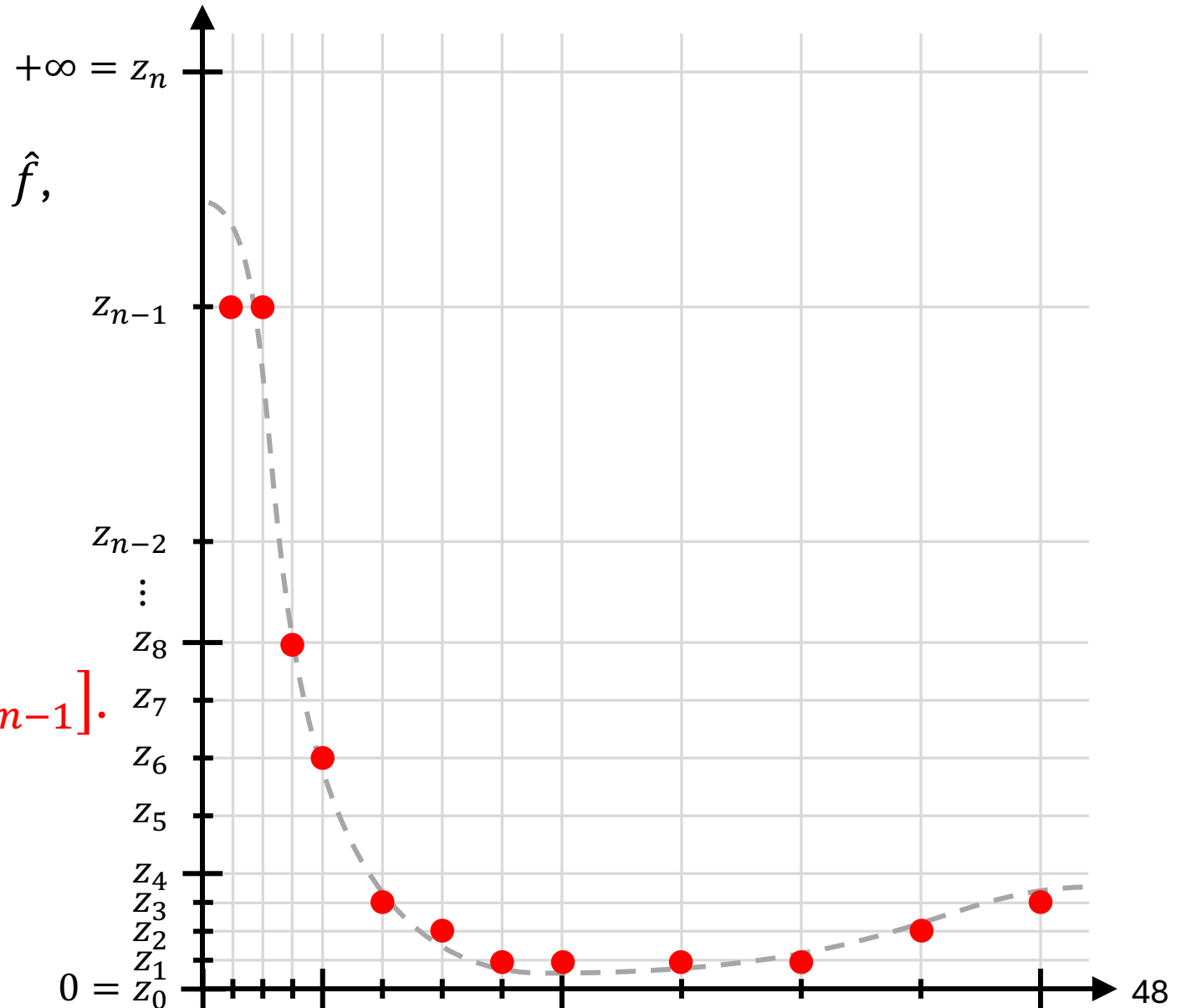


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IUA Theorem Over \mathbb{F}

- **Proof Sketch.**

To approximate the rounded target function \hat{f} , we “stack” indicator functions.

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- **Key Challenge.**

Construct the **indicator functions** using NNs while considering the following:

- NNs: Use affine & activation funcs only.
- Floats: Handle rounding errors & overflows.
- Intervals: Match interval semantics.

IUA Theorem Over \mathbb{F}

- **Proof Sketch.** We construct the **scaled indicator function** $\sigma(c_2) \cdot \mathbb{1}[x \leq z]$ as follows.

