Floating-Point Neural Networks Are Provably Robust Universal Approximators

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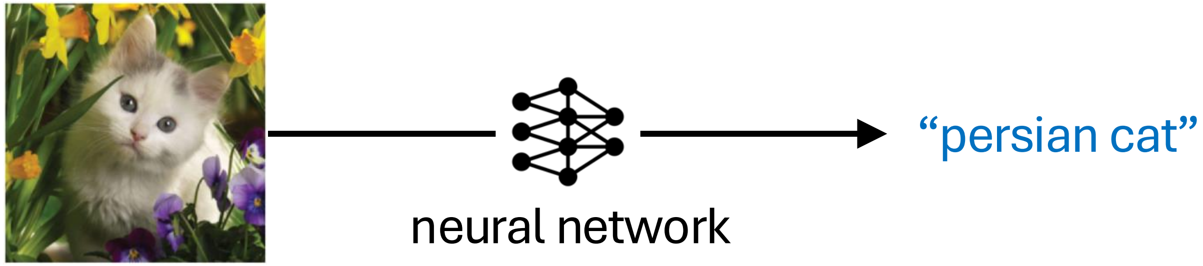
CAV, July 2025

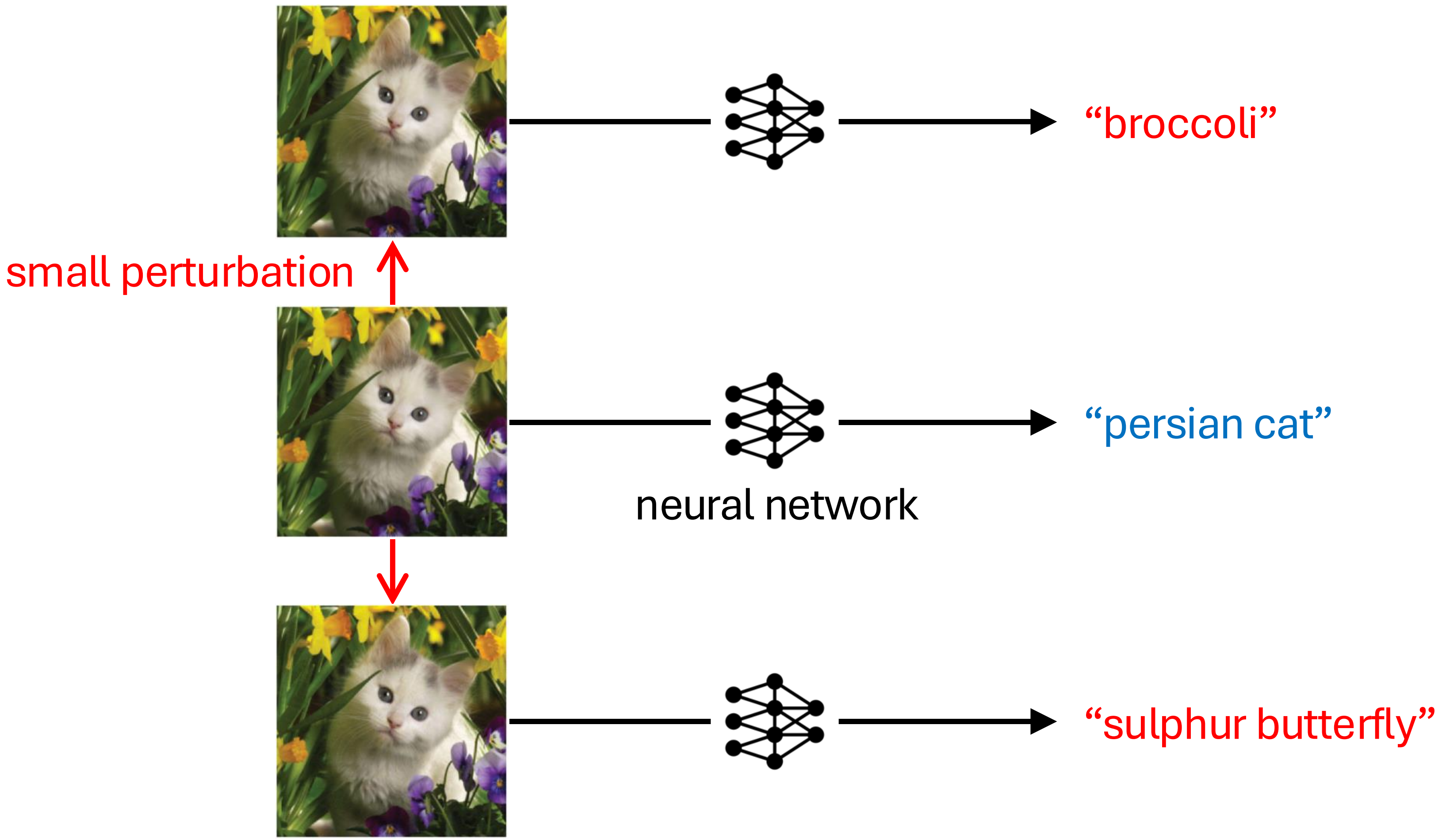
Robustness Issue of Neural Networks

- Neural networks can do amazing things.



Robustness Issue of Neural Networks

- Neural networks can do amazing things. But they are often **not robust.**



Source: arxiv.org/abs/2010.09624

Provably Robust Neural Networks

- Many techniques have been developed to **ensure the robustness** of NNs.
  - Robustness verification: Prove the robustness of a given NN.
  - Robust training: Train a new NN such that it is provably robust (and performs well).

Reluplex: An Efficient SMT Solver for Verifying Deep Neural Networks

Guy Katz, Clark Barrett, David Dill, Kyle Julian and Mykel Kochenderfer

…

Differentiable Abstract Interpretation for Provably Robust Neural Networks

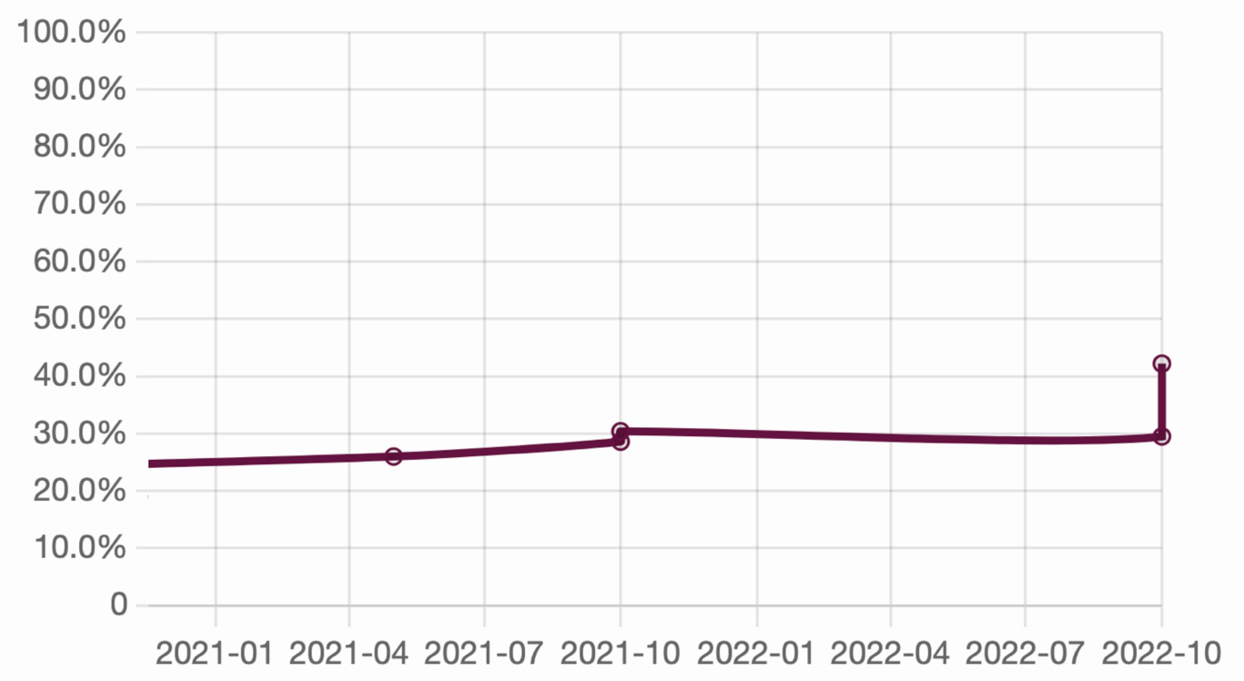
Matthew Mirman [1] Timon Gehr [1] Martin Vechev [1]

…

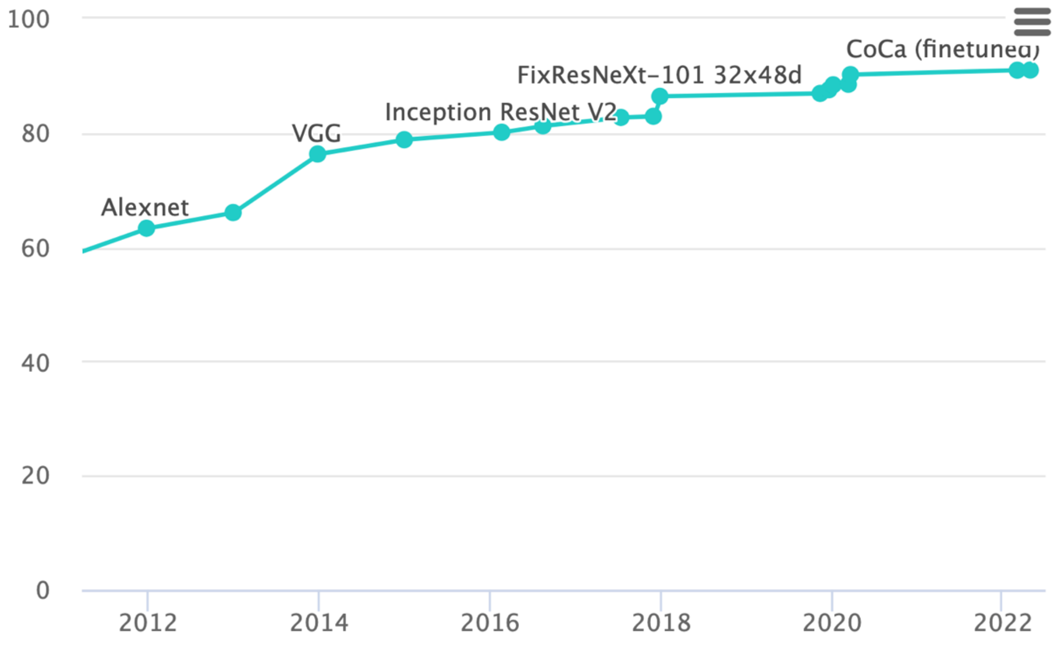
Provably Robust Neural Networks

- **Provably robust** NNs still fail to achieve the state-of-the-art accuracy.

Image classification on ImageNet



provably robust NNs



state-of-the-art NNs
(not provably robust)

Sources: https://sokcertifiedrobustness.github.io/leaderboard/, https://paperswithcode.com/sota/image-classification-on-imagenet

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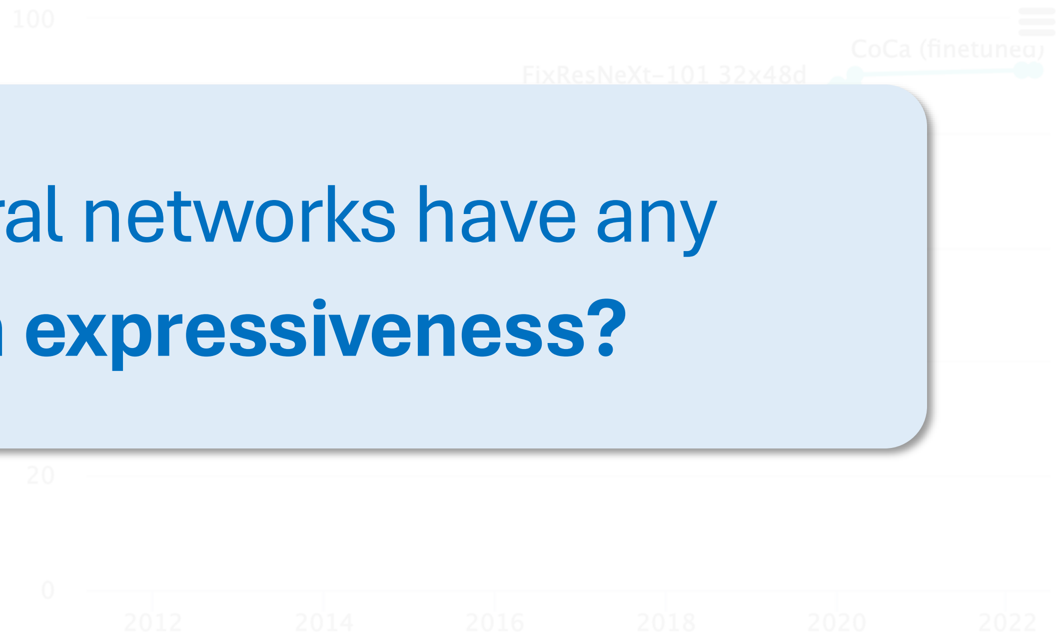

Provably robust NNs

State-of-the-art NNs

Do **provably robust** neural networks have any **fundamental limits** in **expressiveness?**

Sources: https://sokcertifiedrobustness.github.io/leaderboard/, https://paperswithcode.com/sota/image-classification-on-imagenet

Expressiveness of ~~Provably Robust~~ Neural Networks

- No fundamental limit exists by **universal approximation (UA)** theorems.
- **Theorem.** $f : [-1,1]^d \to \mathbb{R}$ $\cdots$ target func (continuous).
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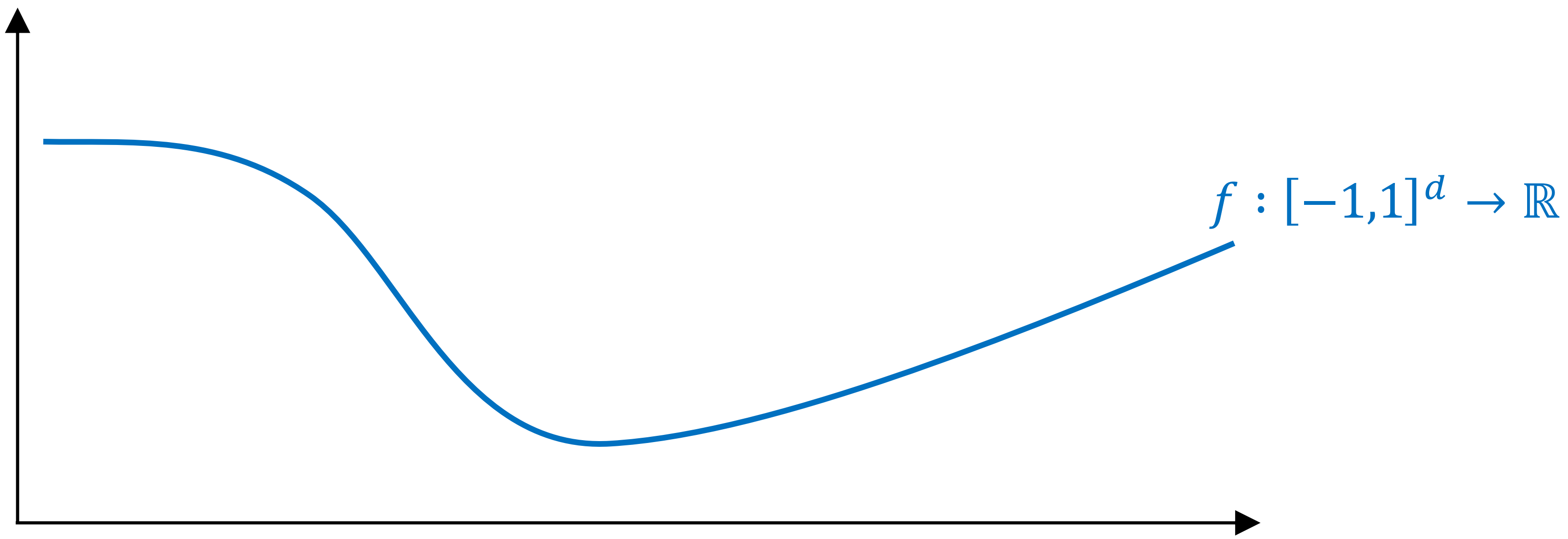

[Cybenko 1989, Pinkus 1999, ...]

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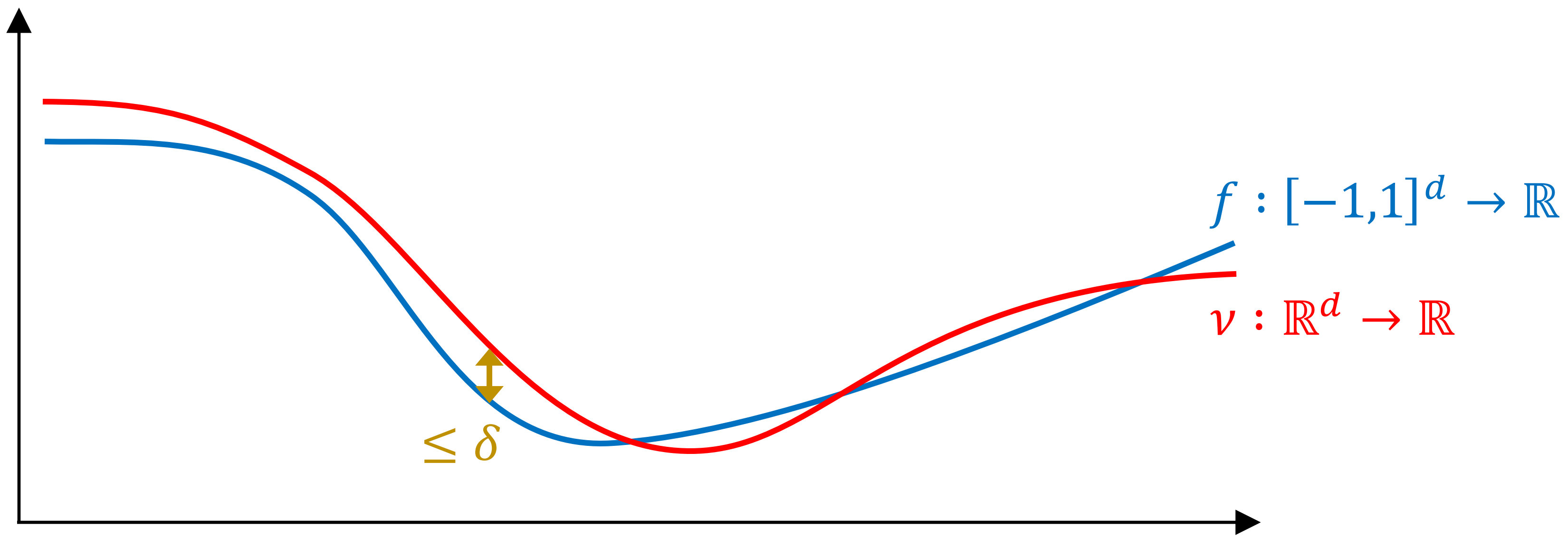
  For any $\delta > 0$, there exists a σ-neural network $\nu : \mathbb{R}^d \to \mathbb{R}$ such that (fully-connected)

$$|\nu(x) - f(x)| \leq \delta \quad \text{for all } x \in [-1,1]^d.$$



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- **Definition.** For $\nu : \mathbb{R}^d \to \mathbb{R}$, define its **interval semantics** as $\nu^\# : \mathrm{Box}(\mathbb{R}^d) \to \mathrm{Box}(\mathbb{R})$.

  ↑ set of d-dim boxes in $\mathbb{R}^d$

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  - Defined using interval arithmetic: $[a, b] +^\# [c, d] \coloneqq [a + c, b + d], \cdots$
  - Overapproximates ν: $\nu(\mathcal{B}) \subseteq \nu^\#(\mathcal{B})$ for all $\mathcal{B} \in \mathrm{Box}(\mathbb{R}^d)$.

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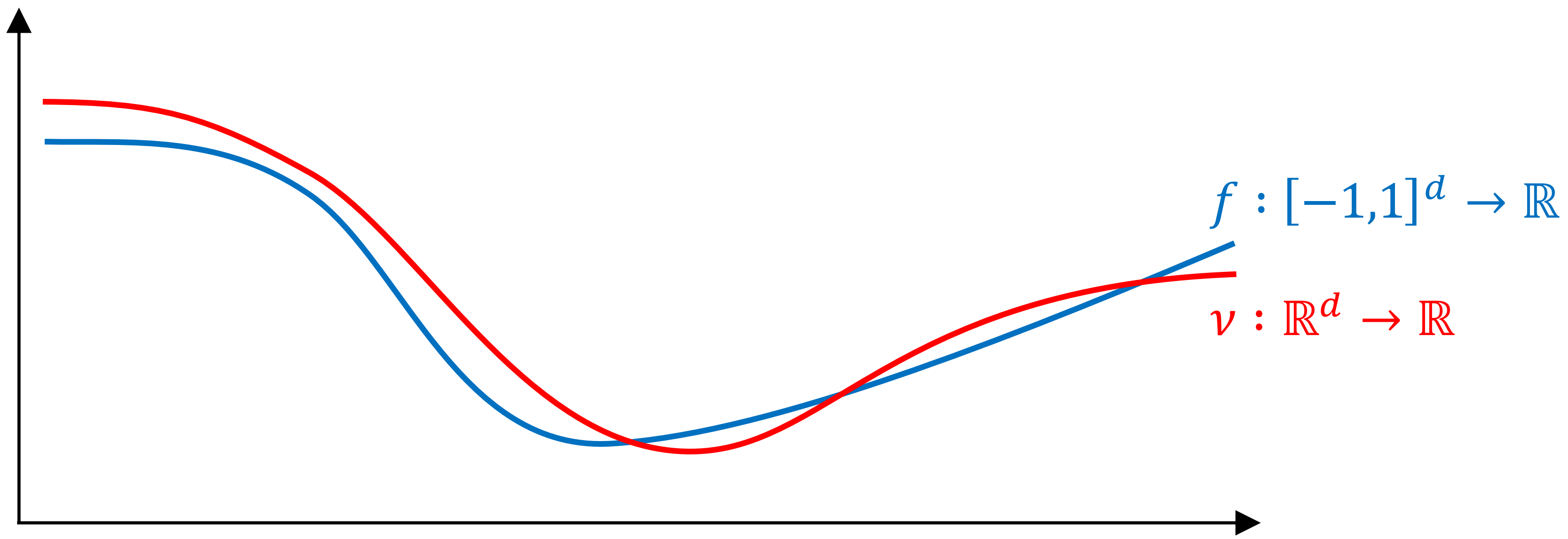

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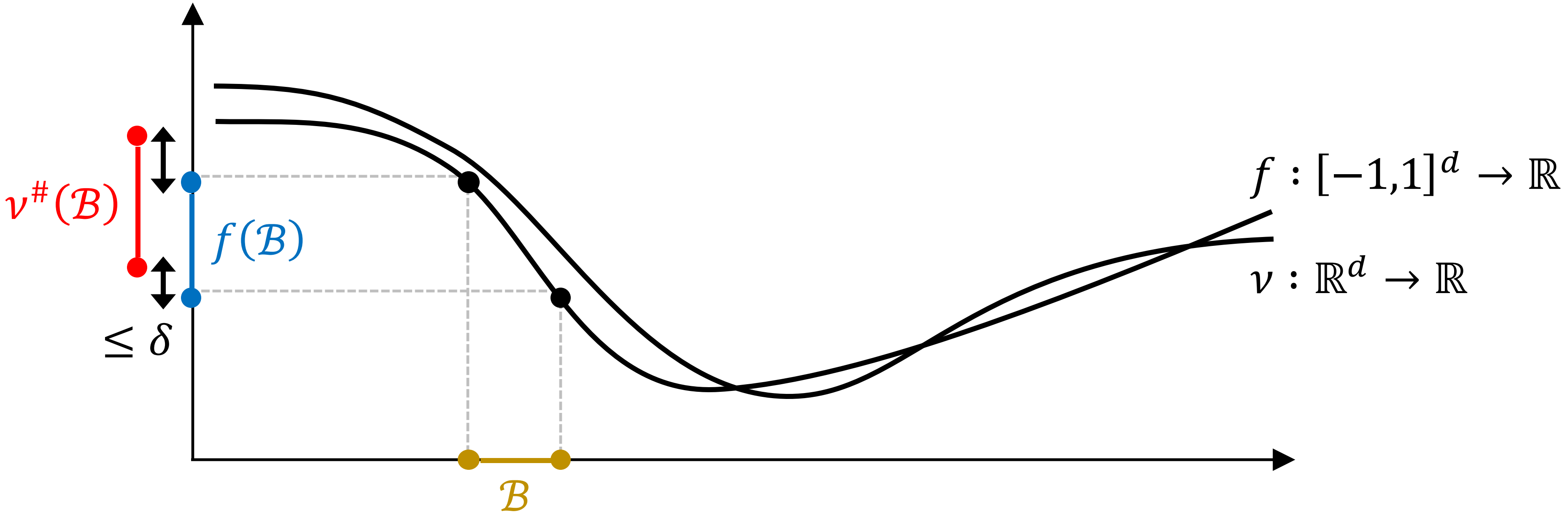

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$\exists$ ideal classifier h (not NN) that is robust (not provably robust)

$\Longrightarrow$ $\exists$ NN that is provably robust and outputs the same as h.

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"Provably robust NNs have no fundamental limits in expressiveness."

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Our Work: Overview

Do existing results still hold in **real-world settings?**

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We study the *expressiveness of provably robust NNs* **over floats.**

- Prove the *IUA theorem* **over floats.**
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- Prove the *computational completeness of "simple" programs* **over floats.**

Our Main Results

What to Prove?

- **Theorem.** Let $f : [-1,1]^d \to \mathbb{R}$ and $\delta > 0$. defined using exact arithmetic
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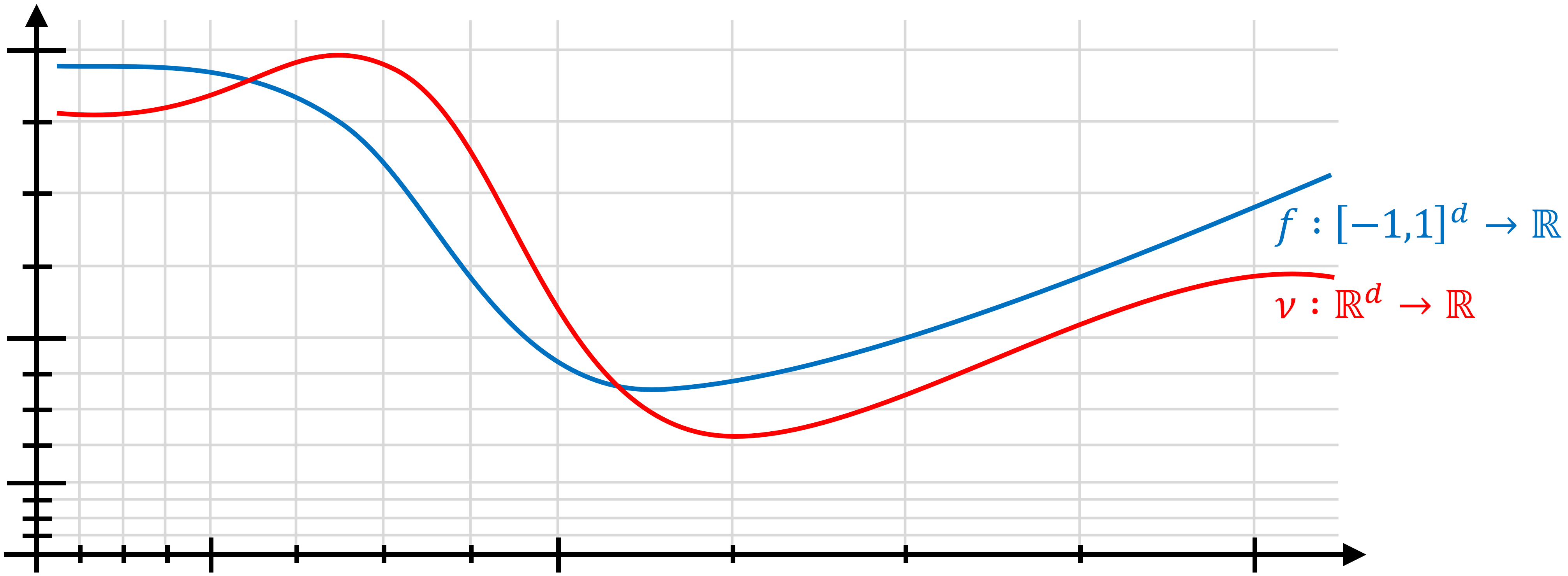

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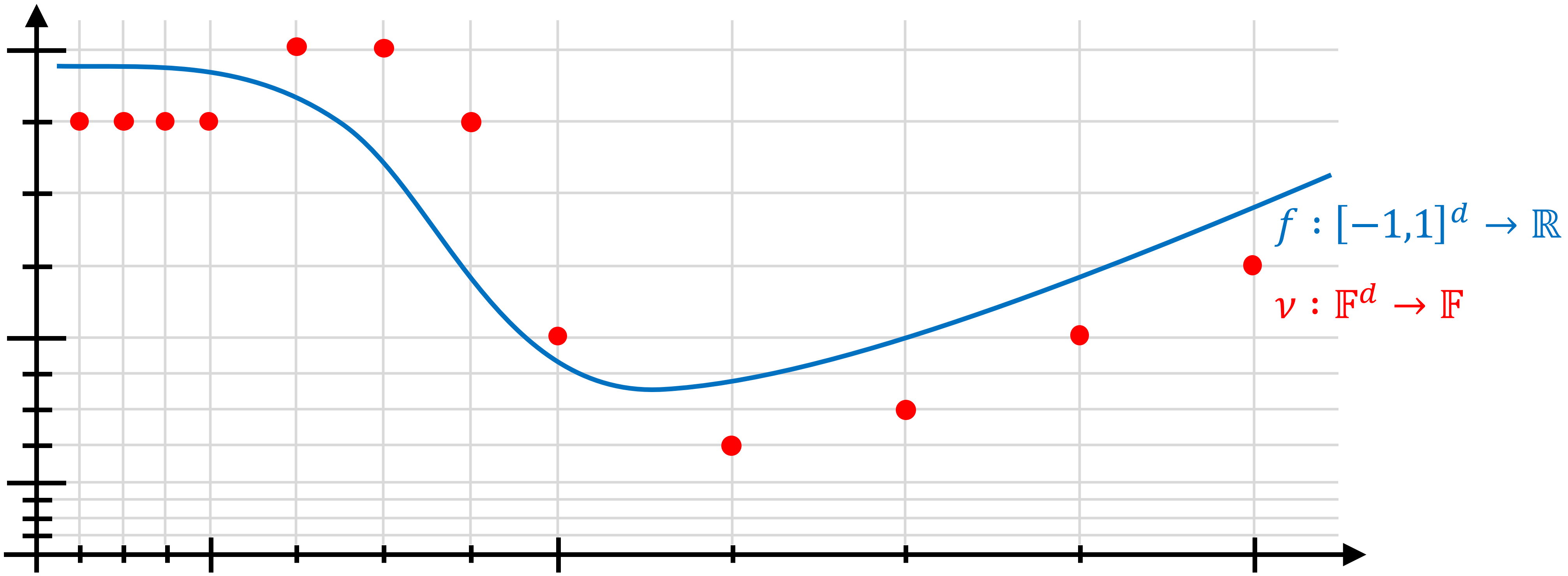

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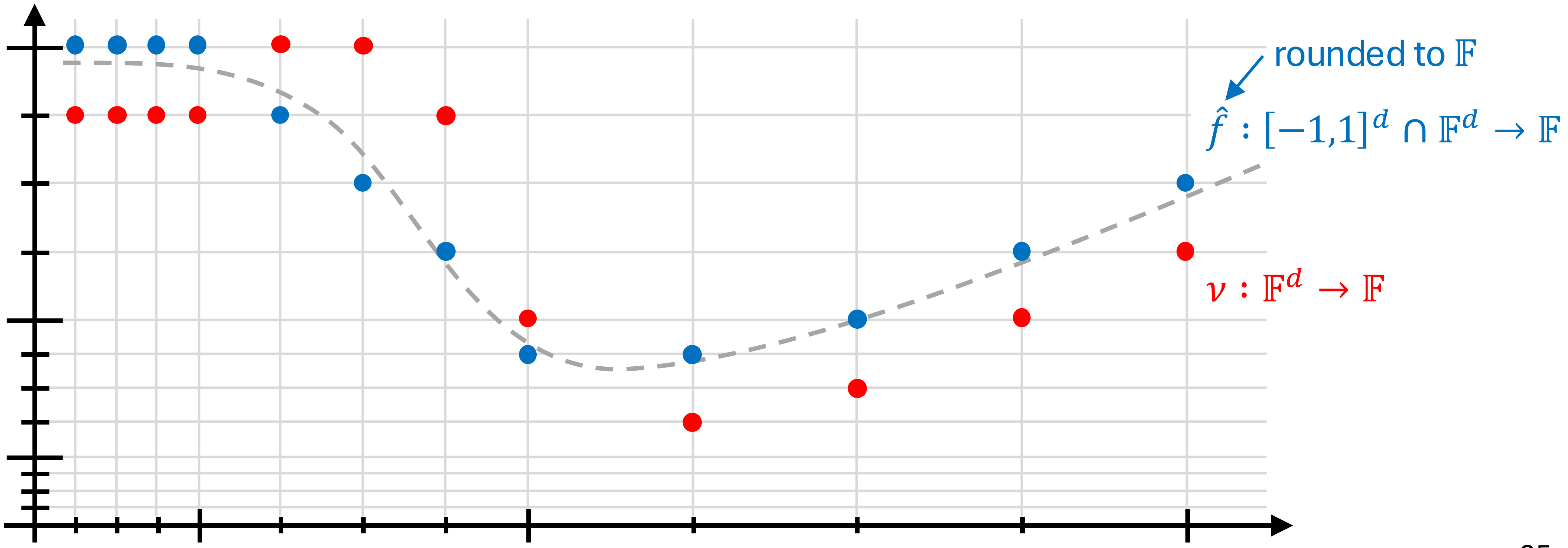

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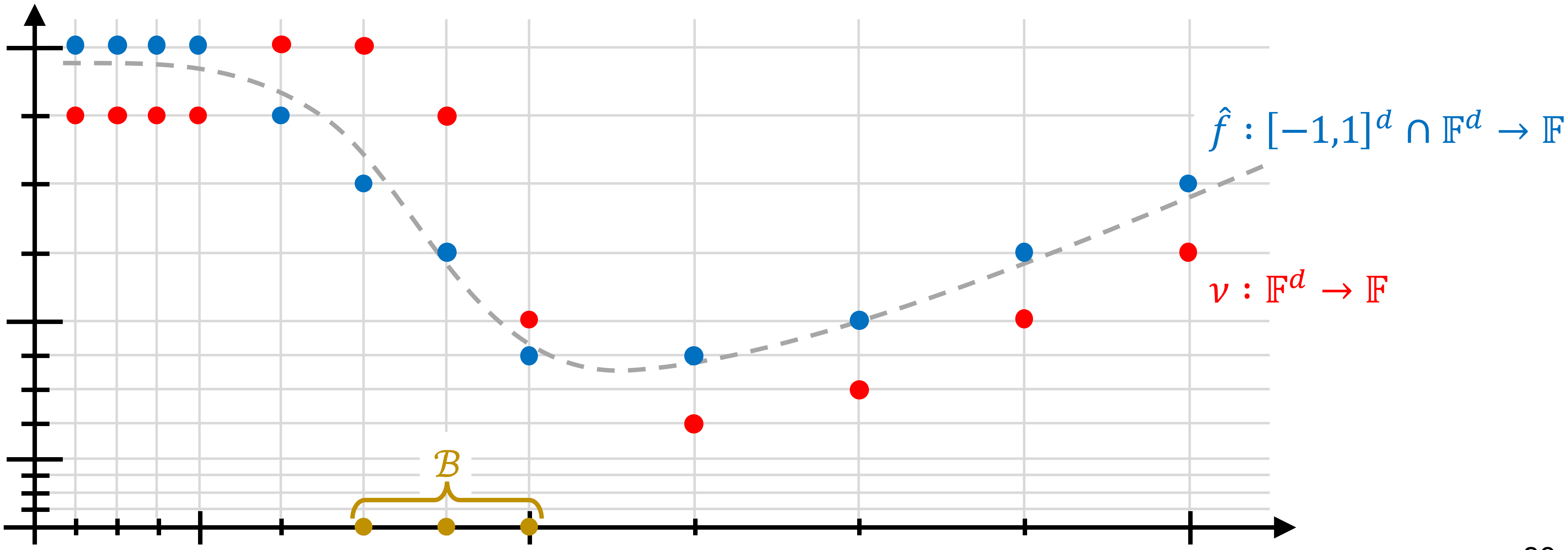

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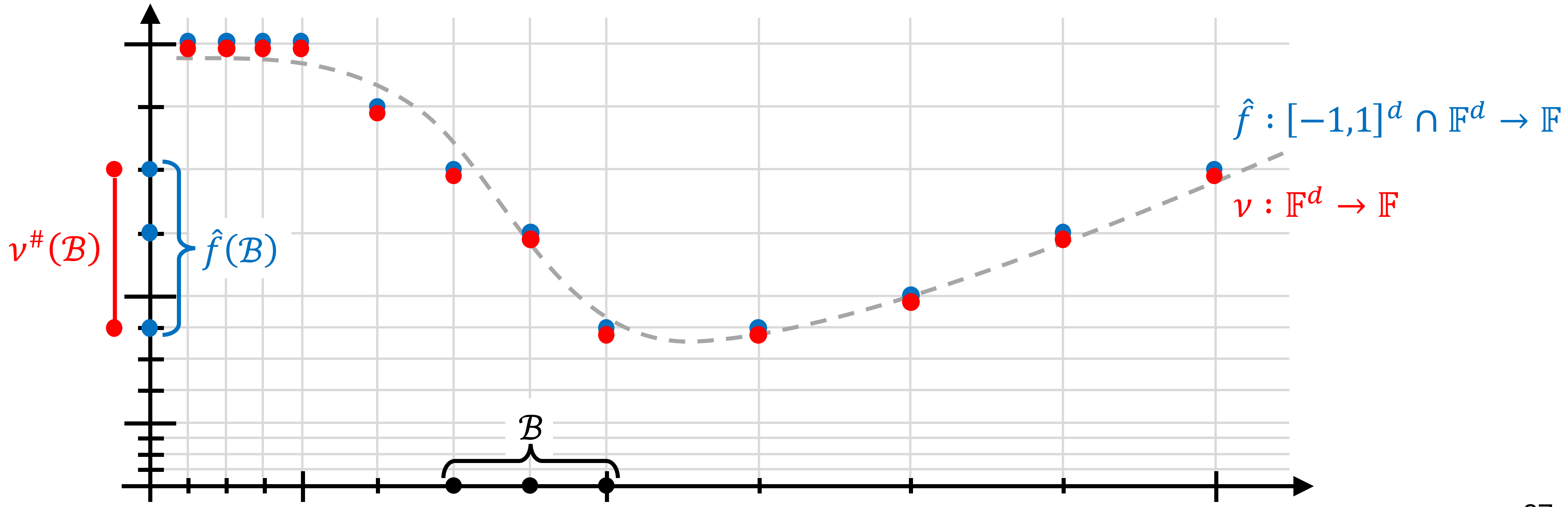

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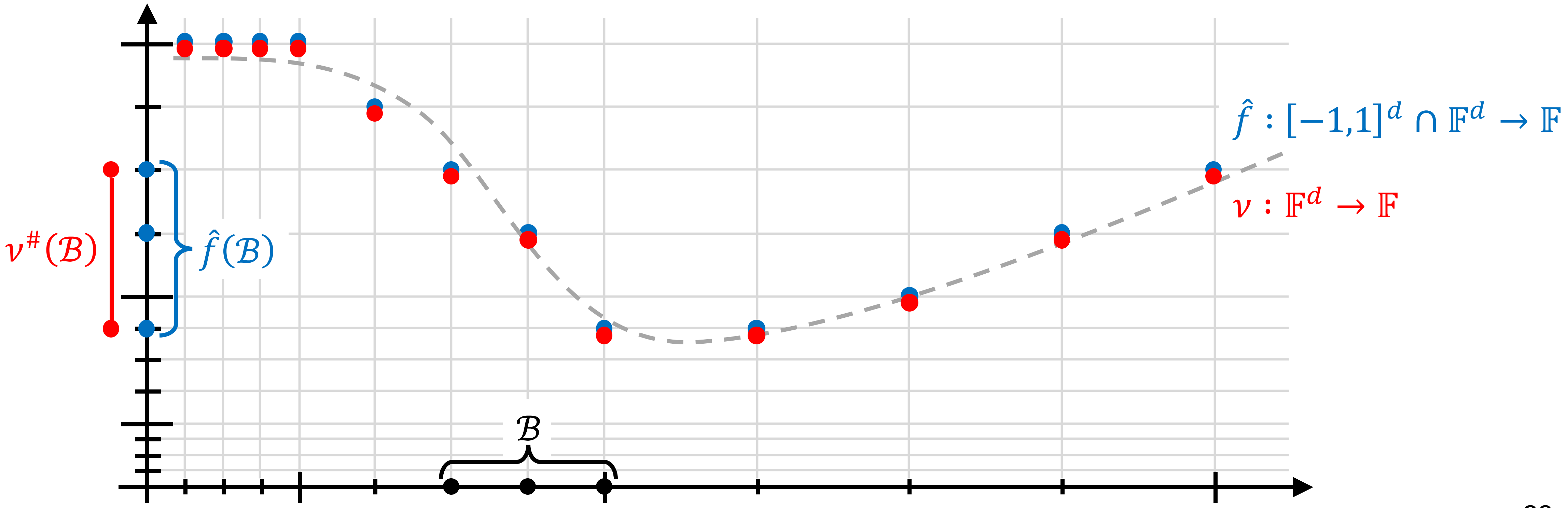

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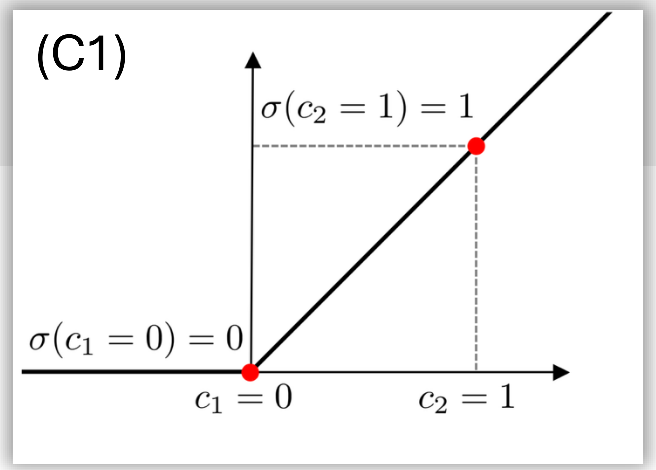

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  $x \le \eta < \eta^+ \le y \implies \sigma(x) \le \sigma(\eta) < \sigma(\eta^+) \le \sigma(y)$
  (or the reverse order holds).

  (C3) $\exists \lambda \in \mathbb{R} \cap [0, 2^{\mathrm{emax}-7}|\sigma(\eta)|]$ such that for all $x, y \in \mathbb{F}$,
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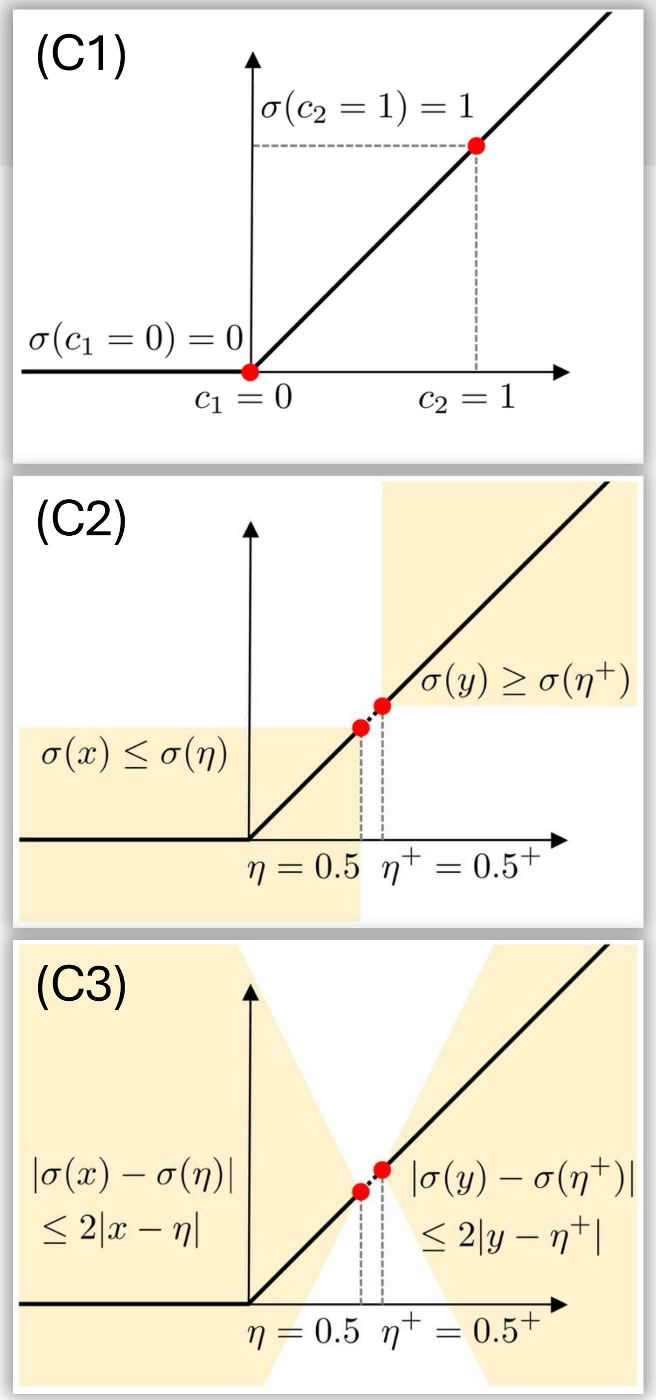

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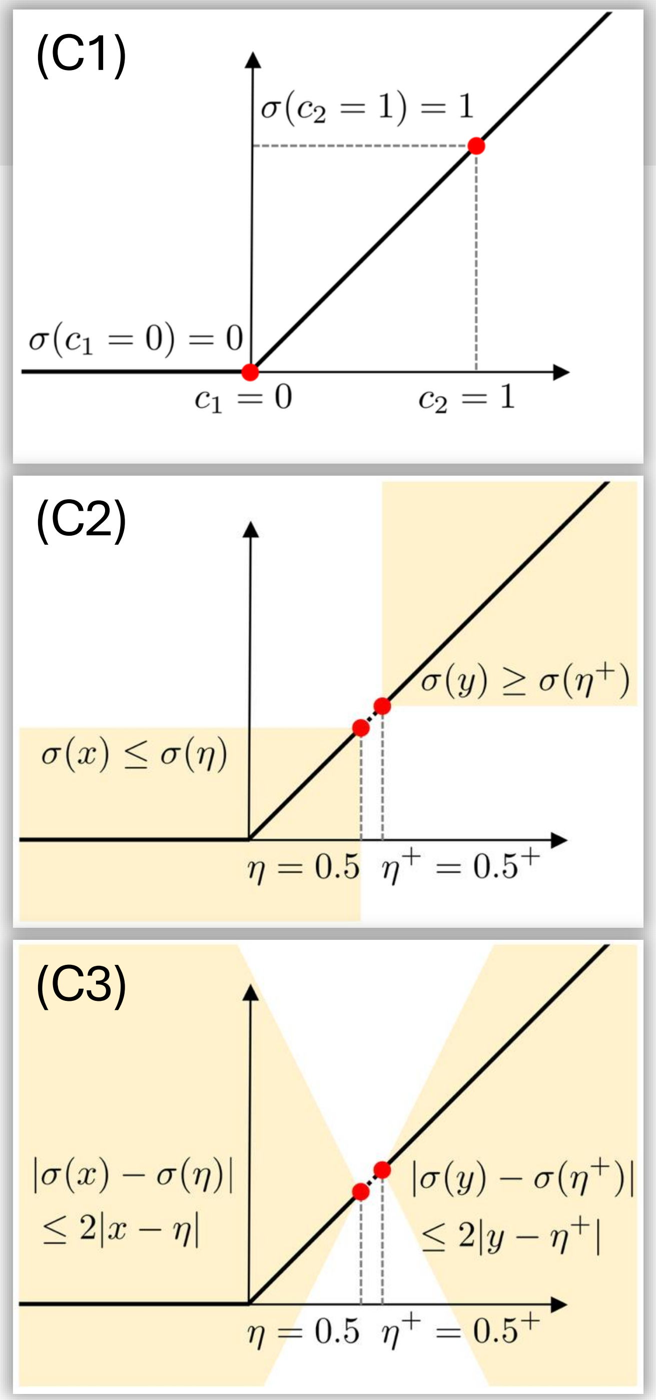
- **Proposition.** The correct roundings of the following activation func's satisfy the conditions (C1)--(C3):

  ReLU, LeakyReLU, ELU, GELU, Mish, softplus, sigmoid, tanh $: \mathbb{R} \to \mathbb{R}$.



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- **Approximation Power.**
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    - σ-NN over $\mathbb{F}$ can be non-affine over $\mathbb{R}$ ($\because$ $\mathrm{aff}_{\mathbb{F}} : \mathbb{F}^k \to \mathbb{F}$ are often non-affine over $\mathbb{R}$ by rounding error).

Implications of Our IUA Theorem

Provable Robustness Over $\mathbb{F}$.

- **Theorem (Informal).** $\exists$ ideal classifier f over $\mathbb{F}$ (not NN) that is robust (not provably robust)
  $\implies$ $\exists$ neural network over $\mathbb{F}$ that is provably robust and outputs the same as f.

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  $\Longrightarrow$ $\exists$ neural network over $\mathbb{F}$ that is provably robust and outputs the same as f.

"Provably robust NNs over $\mathbb{F}$ have no fundamental limits in expressiveness."

- **Note.** Positive answer to the main question raised earlier in this talk.

Implications of Our IUA Theorem

Computational Completeness Over $\mathbb{F}$.

- **Theorem (Informal).** All terminating programs that take and return floats can be expressed by straight-line programs using only $\oplus$ and $\otimes$.

  ($\oplus$ and $\otimes$: $+$ and $\times$ over $\mathbb{F}$)

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- **Theorem (Informal).** All terminating programs that take and return floats can be expressed by straight-line programs using only $\oplus$ and $\otimes$.

"{FP programs with $\oplus$, $\otimes$} is computationally complete for {FP programs that halt}."

- **Note.** Important contribution to the FP literature, independent of the NN/verification literature.

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- **Theorem (Informal).** All terminating programs that take and return floats can be expressed by straight-line programs using only $\oplus$ and $\otimes$.

"{FP programs with $\oplus$, $\otimes$} is computationally complete for {FP programs that halt}."

- **Note.** Important contribution to the FP literature, independent of the NN/verification literature.
  Prove this theorem by extending our IUA theorem for $\sigma = \mathrm{id}$.

Summary

Provably robust NNs have no fundamental limit in expressiveness, even over floats.

- Prove the IUA theorem for NNs over $\mathbb{F}$.
- Prove the existence of provably robust NNs over $\mathbb{F}$.

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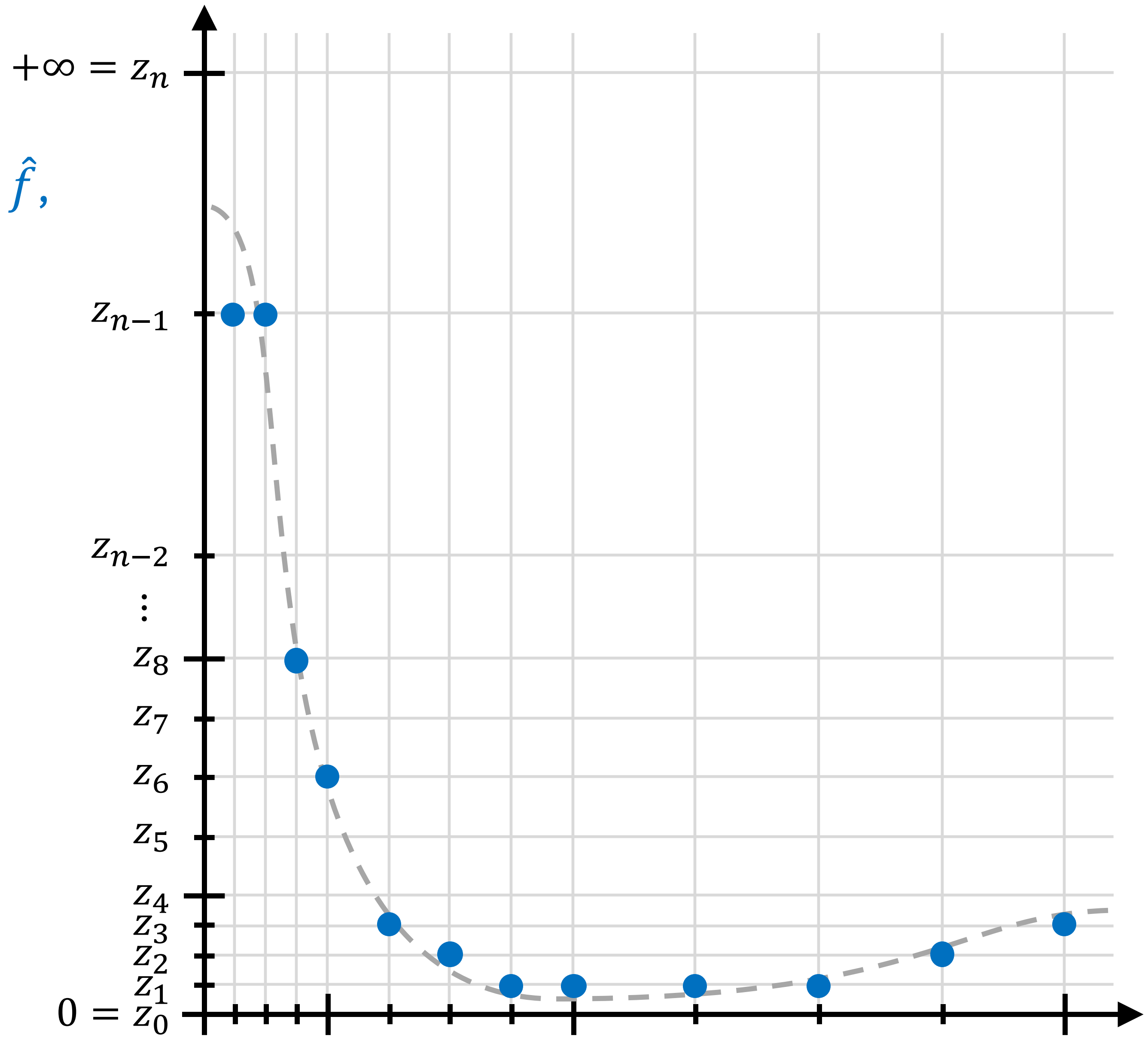
Unexpected byproducts.

- Identify fundamental distinctions between two computations models: over $\mathbb{F}$ and over $\mathbb{R}$.
- Prove that all halting programs over $\mathbb{F}$ can be expressed using only two operations: $\oplus$ and $\otimes$.

IUA Theorem Over $\mathbb{F}$

- **Proof Sketch.**

  To approximate the rounded target function $\hat{f}$, we "stack" indicator functions.



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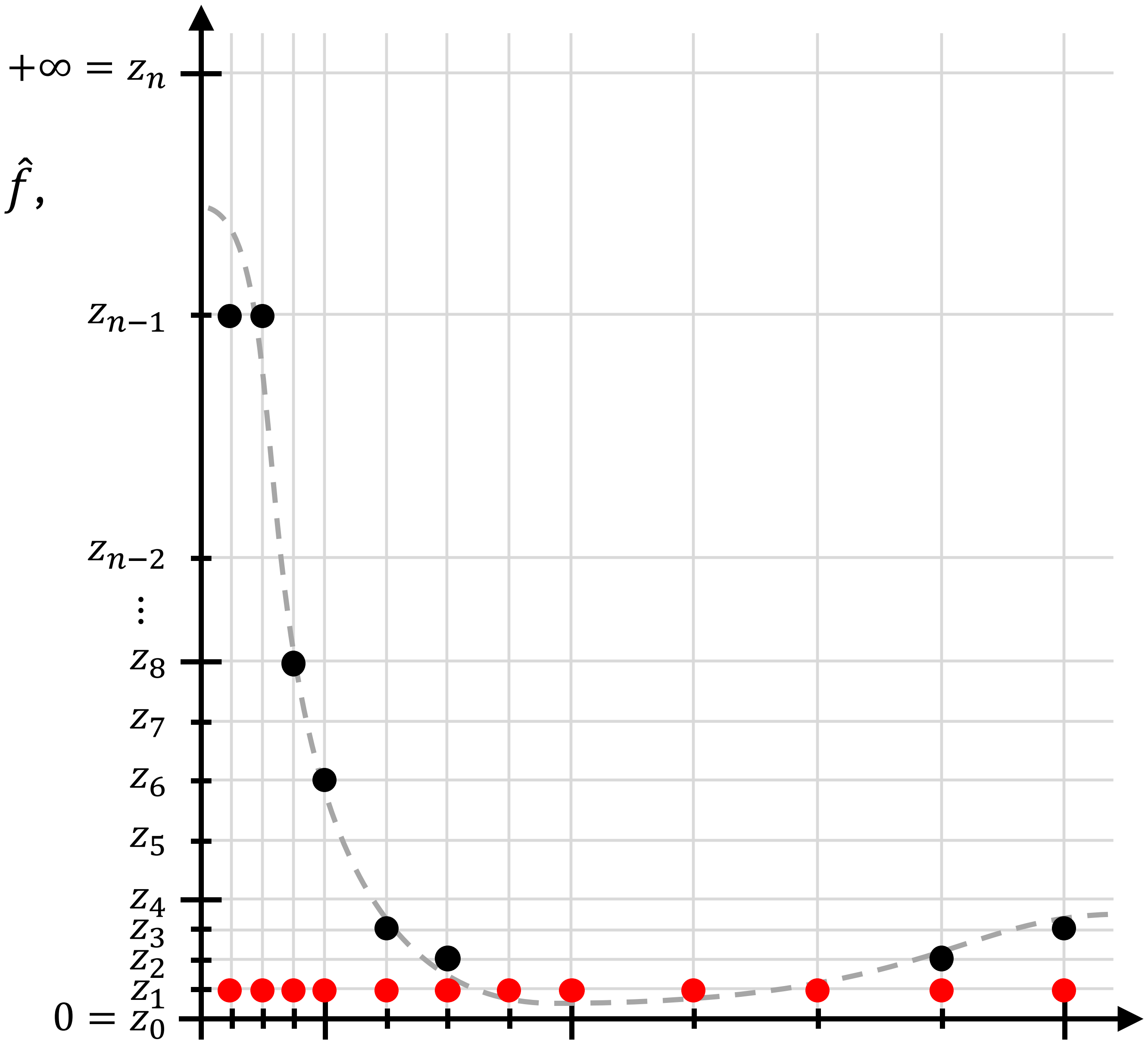

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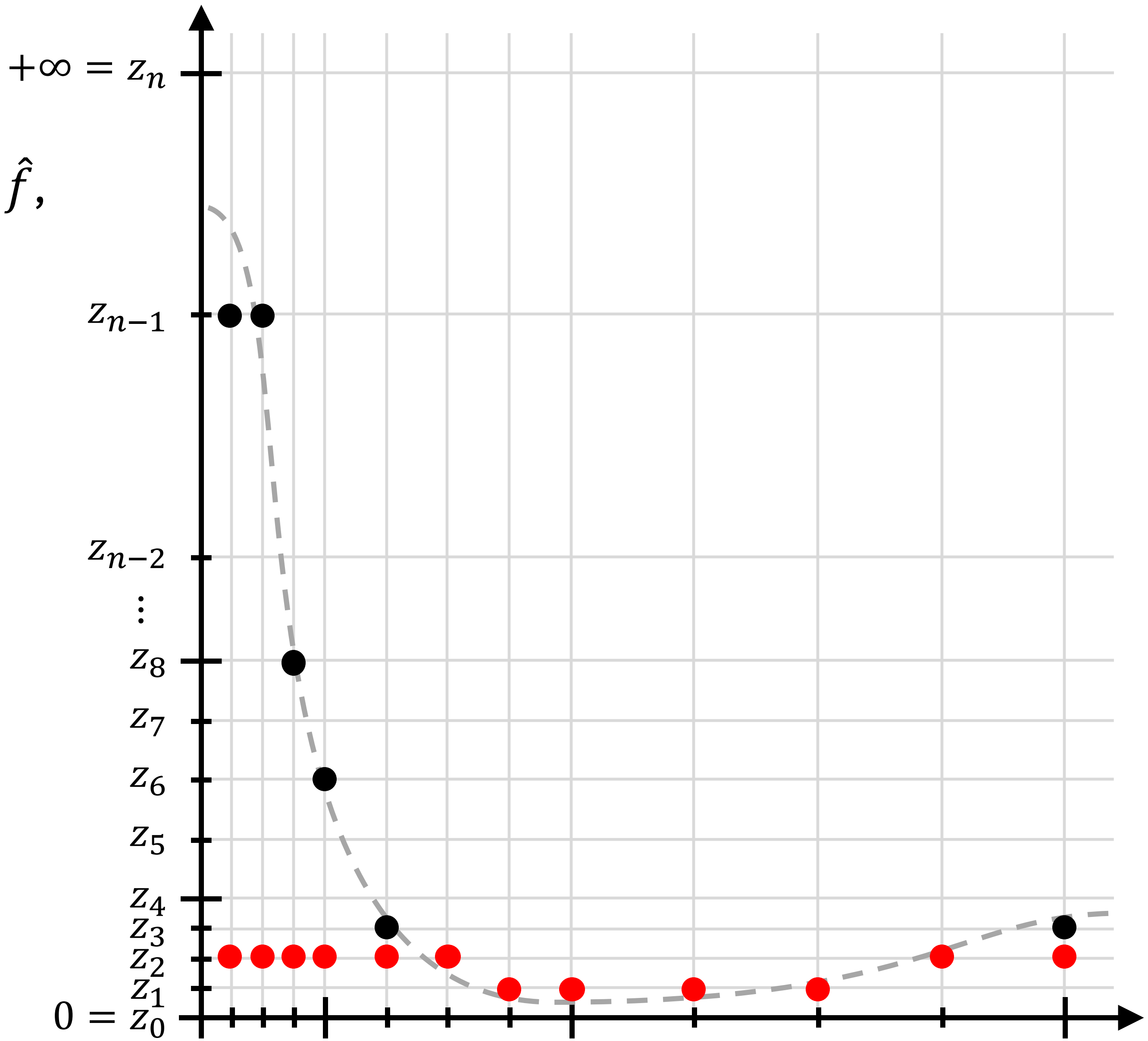

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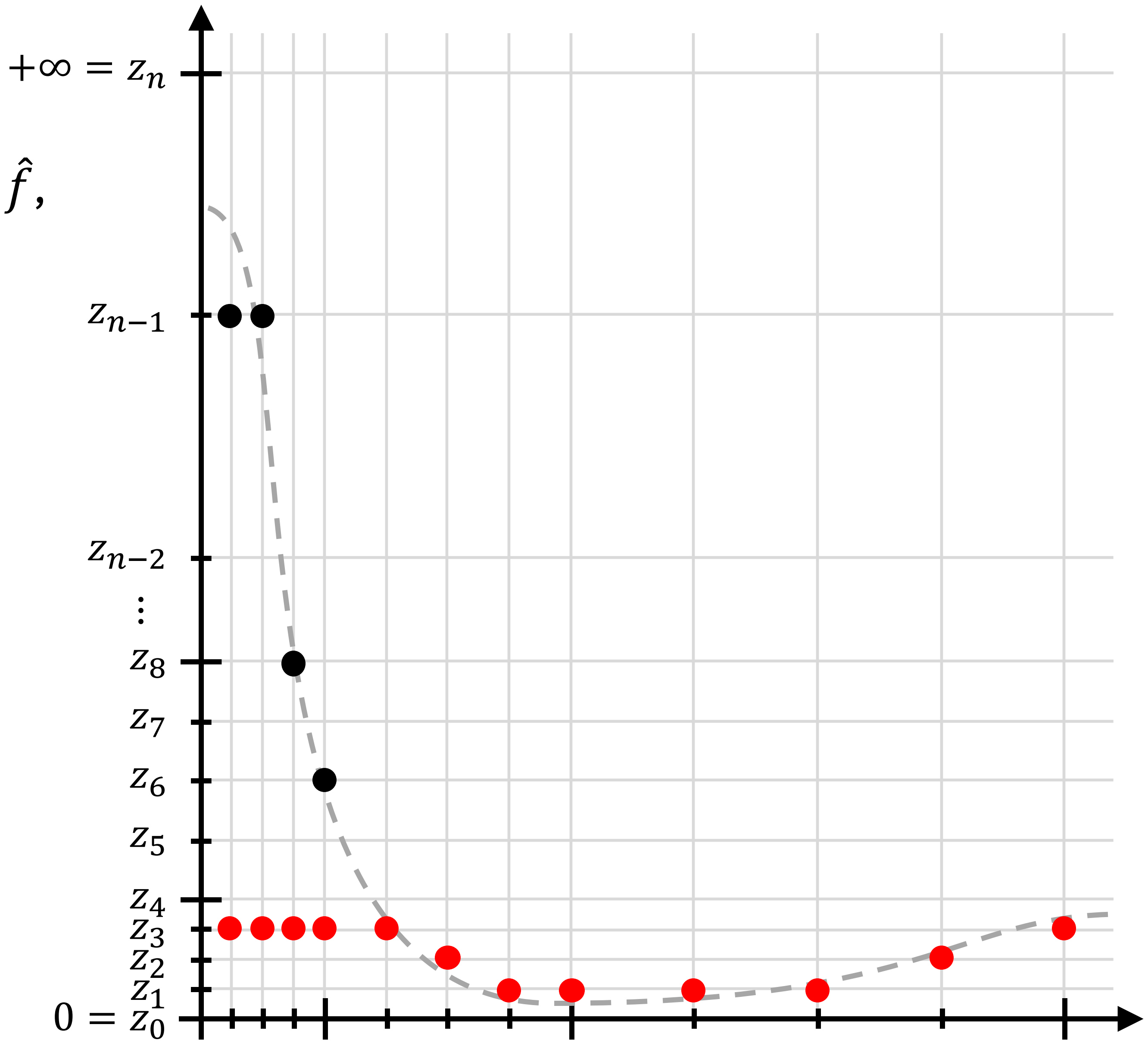
$$



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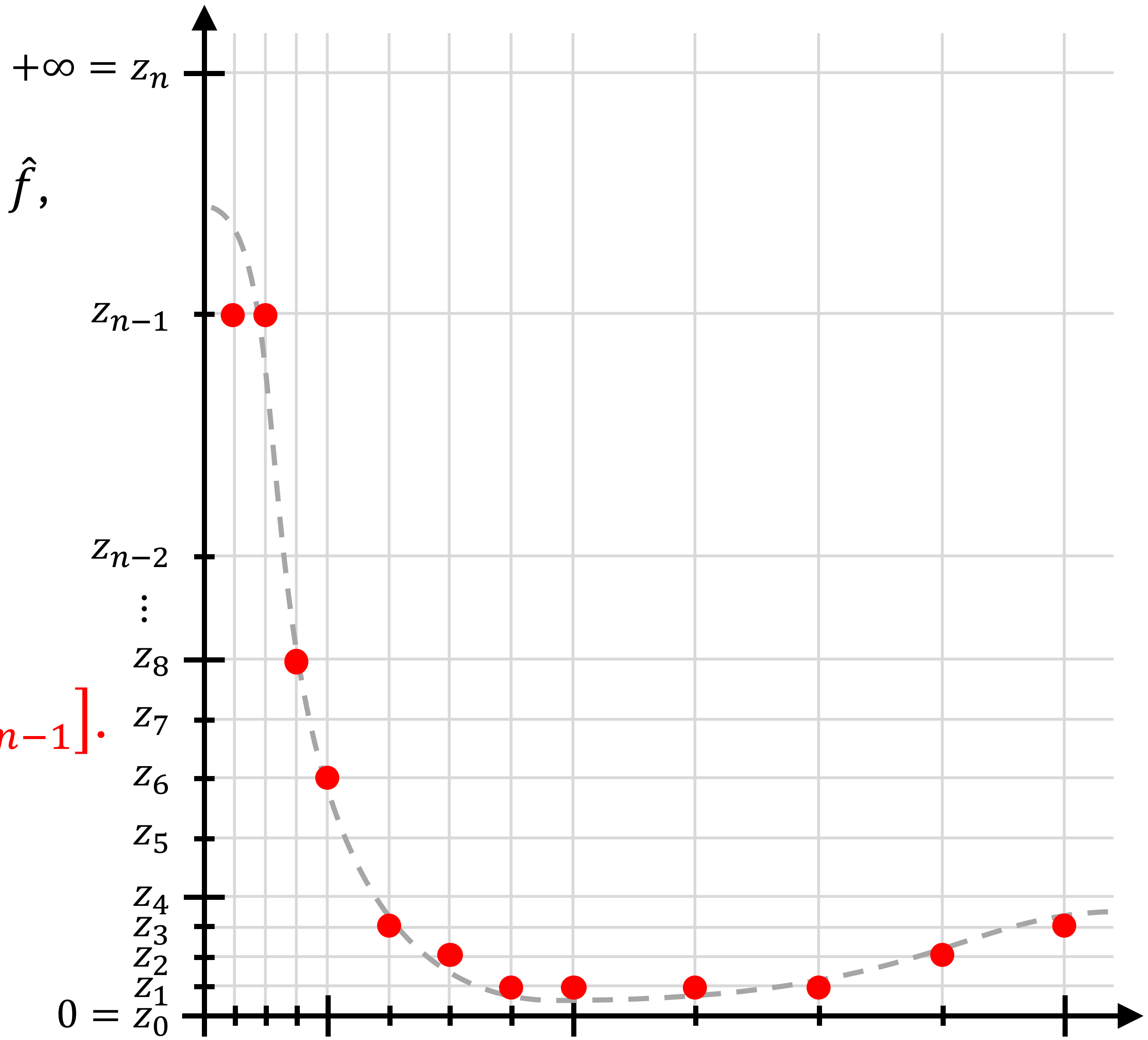

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\end{aligned}
$$

- **Key Challenge.**

Construct the indicator functions using NNs while considering the following:

  - NNs: Use affine & activation funcs only.
  - Floats: Handle rounding errors & overflows.
  - Intervals: Match interval semantics.

IUA Theorem Over $\mathbb{F}$

- **Proof Sketch.** We construct the scaled indicator function $\sigma(c_2) \cdot \mathbb{1}[x \leq z]$ as follows.